AN NIP-LIKE NOTION IN ABSTRACT ELEMENTARY CLASSES

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ABSTRACT. This paper is a contribution to "neo-stability" type of result for abstract elementary classes. Under certain set theoretic assumptions, we propose a definition and a characterization of NIP in AECs. The class of AECs with NIP properly contains the class of stable AECs [1]. We show that for an AEC K and $\lambda \geq LS(K)$, K_λ is NIP if and only if there is a notion of nonforking on it which we call a w*-good frame. On the other hand, the negation of NIP leads to being able to encode subsets.

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1. INTRODUCTION

There is a massive body of literature on "neostability" for first order theories dedicated to exploration and study of forking-like relations for various classes of unstable theories. The main classes: NIP theories, simple theories, theories with the strict order property, theories with the tree property of type 1 and 2, were all presented by Shelah in [She78]. In mid 1976 Shelah set the program which he named **classification theory for non-elementary classes**. A few years later the focus shifted to abstract elementary classes (AECs).

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[1]See Examples 2.20 and 2.21 for AECs that are unstable, not elementary but NIP.

An appropriate generalization of stability for AECs was introduced in [She99] building on many previous papers including [She71b] and [GS]. In the last forty years starting with [GS86] much was discovered about analogues of superstability. See [Vas16b], [GV17], and [Leu23] for some recent work.

In this paper we propose progress towards "neostability of AECs", more precisely, exploring an analogue of NIP and its negation. We propose a definition (under a certain cardinal arithmetic axiom) of NIP. Using techniques from papers by Shelah [She09a], Jarden and Shelah [JS13] and Mazari-Armida [MA20], we obtain a characterization of NIP in AECs using frames (a forking-like relation).

The notion of the λ-stable AEC was first studied in [She99] using non-splitting. Various frameworks of forking-like relations were introduced. In [She09a], Shelah introduced the local notion of the good λ-frame, an axiomatization of forking-like relations for structures of cardinality λ in AECs, as a parallel of superstability. In [BG17] Boney and Grossberg established that for "nice" AECs, stablity implies existence of strong independence relations on the subclass of saturated models, which allows types of arbitrary length. In [BGKV16] it was shown that this relation and several others are unique/canonical (if they exist).

Although good λ-frames are nice and powerful, sometimes they might not exist. There are several weaker notions, where some of the axioms of a good λ-frame are weakened or dropped. Vasey worked with good$^-$ λ-frames in [Vas16b] and good^{-S} λ-frames in [Vas16a]. Jarden and Shelah defined semi-good λ-frames in [JS13]. Mazari-Armida introduced w-good λ-frames in [MA20], which is weaker than all the axiomatic frames mentioned above.

Definition 1.1. Let K be an AEC, $\lambda \geq LS(K)$. K_λ has NIP if for all $M \in K_\lambda$, $|gS(M)| \leq \text{ded } \lambda$.

Our definition of NIP will be discussed further in the next section.

Our main results are:

Theorem 1.2 ($2^{\lambda^+} > 2^\lambda$)**.** Let K be an AEC with $\lambda \geq LS(K)$ with λ-AP, λ-JEP and λ-NMM, and $1 \leq I(\lambda^+, K) < 2^{\lambda^+}$. K_λ has NIP if and only if there is a w*-good λ-frame on K except possibly without (Continuity$^-$). Moreover,

(1) ($\text{ded } \lambda = \lambda^+ < 2^\lambda$) If $\mathfrak{s}_{\lambda-unq}$ satisfies in addition (Continuity), then the w*-good frame satisfies in addition that if $p \in S^{bs}(M)$, then there is $N \geq_K M$ and $q \in S^{bs}(N)$ extending p that does not fork over N. In particular, for any $N' \geq_K N$ there is $q' \in gS(N')$ extending q that does not fork over N.
(2) if K is $(< \lambda^+, \lambda)$-local, then the frame has (Continuity$^-$).

Theorem 1.3. Suppose K is $(< \aleph_0)$-tame, $M \in K$, $C \subseteq |M|$, $\lambda := \|C\| \geq \beth_3(LS(K))$ and $(\text{ded } \lambda)^{2^{LS(K)}} = \text{ded } \lambda$. Suppose $|gS^1(C; M)| > \text{ded } \lambda$. Then there is $N \in K$, $\langle \bar{a}_n \in^m |N| \mid n < \omega \rangle$ and ϕ in the language of Galois Morleyization such that for every $n < \omega$ and $w \subseteq n$ there is $b_w \in |N|$ such that for all $i < n$,

$$N \models \phi(\bar{a}_i, b_w) \iff i \in w$$

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It is interesting to comment that Shelah already implicitly discussed similar results in [She01] dealing with Grossberg's question "Does $I(\lambda, K) = I(\lambda^{++}, K) = 1$ imply $K_{\lambda^{++}} \neq \emptyset$" and in its followup [She09a], Chapter II of [She09c], and [She09b], Chapter VI of [She09d]. More specifically, in [She09d, VI.2.3] and [She09d, VI.2.5] Shelah considered the number of branches of a tree as a bound of Galois types over a model.

2. Preliminaries

Notation 2.1.

(1) For any structure M in some language, we denote its universe by $|M|$, and its cardinality by $\|M\|$.
(2) For cardinals λ and μ, $[\lambda, \mu) := \{\kappa \in \text{Card} \mid \lambda \leq \kappa < \mu\}$. $[\lambda, \infty) := \{\kappa \in \text{Card} \mid \lambda \leq \kappa\}$.
(3) $K_{[\lambda,\mu)} := \{M \in K \mid \|M\| \in [\lambda, \mu)\}$. $K_\lambda := K_{[\lambda,\lambda^+)}$

Definition 2.2. For K an AEC, we say:

(1) K has the amalgamation property (AP) if for all $M_0 \leq M_l$ for $\ell = 1, 2$, there is $N \in K$ and K-embeddings $f_\ell : M_\ell \to N$ for $\ell = 1, 2$ such that $f_1 \restriction_{M_0} = f_2 \restriction_{M_0}$.
(2) K has the joint embedding property (JEP) if for all M_0, $M_1 \in K$ there are $N \in K$ and K-embeddings $f_\ell : M_l \to N$ for $\ell = 0, 1$.
(3) K has no maximal models (NMM) if for all $M \in K$ there is $N >_K M$.

Remark 2.3. For a property P, e.g. amalgamation, we say that K_λ has P or that K has λ-P if we restrict to K_λ in the above definition.

Definition 2.4.

(1) $K^3_\lambda := \{(a, M, N) \mid M, N \in K_\lambda, M <_K N, a \in |N| - |M|\}$.
(2) For (a_0, M_0, N_0), $(a_1, M_1, N_1) \in K^3_\lambda$, $(a_0, M_0, N_0) \leq (a_1, M_1, N_1)$ if $M_0 \leq M_1$, $a_0 = a_1$ and $N_0 \leq_K N_1$.

(3) For (a_0, M_0, N_0), $(a_1, M_1, N_1) \in K^3_\lambda$ and K-embedding $h : N_0 \to N_1$, $(a_0, M_0, N_0) \leq_h (a_1, M_1, N_1)$ if $h \restriction_{M_0}: M_0 \to M_1$ is a K-embedding and $h(a_0) = a_1$.

Definition 2.5.

(1) For $(a_0, M_0, N_0), (a_1, M_1, N_1) \in K^3_\lambda$, $(a_0, M_0, N_0)E_{at}(a_1, M_1, N_1)$ if there are $N \in K$, $f_0 : N_0 \to N$, and $f_1 : N_1 \to N$ K-embeddings such that $f_0(a_0) = f_1(a_1)$ and $f_0 \restriction_M = f_1 \restriction_M$.
(2) E is the transitive closure of E_{at}.
(3) For $(a, M, N) \in K^3_\lambda$, *the Galois type of a over M in N* is $\mathbf{gtp}(a/M, N) := [(a, M, N)]_E$.
(4) For $M \in K_\lambda$, $gS(M) := \{\mathbf{gtp}(a/M, N) \mid (a, M, N) \in K^3_\lambda\}$.

Remark 2.6. If K_λ has AP then $E_{at} = E$.

Definition 2.7. Assume that K_λ has AP. For M, $N \in K$, $p \in gS(M)$ and K-embedding $h : M \to N$, $h(p) := \mathbf{gtp}(h'(a)/h[M], N)$, where $h' : M' \to N'$ extends h and $(a, M, M') \in p$. Note that $h(p)$ does not depend on the choice of (a, M, M') or h'. See [Leu23, 3.1] for a proof.

Definition 2.8. Let $\langle M_i \mid i < \delta\rangle$ be increasing continuous. A sequence of types $\langle p_i \in S(M_i) \mid i < \delta\rangle$ is coherent if there are (a_i, N_i) for $i < \delta$ and $f_{j,i} : N_j \to N_i$ for $j < i < \delta$ such that:

(1) $f_{k,i} = f_{j,i} \circ f_{k,j}$ for all $k < j < i$.
(2) $\mathbf{gtp}(a_i/M_i, N_i) = p_i$.
(3) $f_{j,i} \restriction_{M_j} = id_{M_j}$.
(4) $f_{j,i}(a_j) = a_i$.

The notion of coherent sequence of types first appeared in [GV06, 2.12], Here we use the version in [MA20, 3.14] that avoids the use of a monster model.

Fact 2.9. Let δ be a limit ordinal and $\langle M_i \in K \mid i \leq \delta\rangle$ increasing continuous, and $\langle p_i \in gS(M_i) \mid i < \delta\rangle$ a coherent sequence of types. Then there is $p \in gS(M_\delta)$ an upper bound of $\langle p_i \in gS(M_i) \mid i < \delta\rangle$.

Fact 2.10. [Bal09, 11.3(2)] Let δ be a limit ordinal, $\langle M_i \in K \mid i \leq \delta\rangle$ increasing continuous, and $\langle p_i \in gS(M_i) \mid i < \delta\rangle$ a sequence of types with upper bound $p \in S(M_\delta)$. Then there are $\langle N_i \mid i \leq \delta$ and $\langle f_{j,i} \mid j < i\rangle$ that witness $\langle p_i \in gS(M_i) \mid i \leq \delta\rangle$ a coherent sequence.

Definition 2.11. [She01, 0.22(2)] Let $\mu > \lambda$. $N \in K_\mu$ is *saturated in μ over λ* if for all $M \leq_K N$, $\lambda \leq \|M\| < \mu$, N realizes $gS(M)$.

Definition 2.12. [She01, 0.26(1)] Let $\mu > \lambda$. $N \in K_\mu$ is *homogeneous in μ for λ* if for all $M_1 \leq_K N$, $M_1 \leq_K M_2 \in K_\lambda$, $\lambda \leq \|M_1\| \leq \|M_2\| < \mu$, there is K-embedding $f : M_2 \to N$ over M_1.

Fact 2.13. [She01, 0.26(1)] Let $\mu > \lambda$. If K_λ has AP then $M \in K_\mu$ is saturated over μ for λ if and only if M is homogeneous over μ for λ.

Definition 2.14. [She71a] For a cardinal λ,

$$\text{ded } \lambda := \sup\{\kappa \mid \exists \text{ a regular } \mu \text{ and a tree } T \text{ with } \leq \lambda \text{ nodes and } \kappa \text{ branches of length } \mu, |T| = \kappa\}.$$

Fact 2.15. [She78, II.4.11] Let T be a complete first order theory and ϕ a formula in its language. λ is an infinite cardinal such that $2^\lambda > \text{ded } \lambda$. The following are equivalent:

(1) ϕ has the independence property.
(2) $|S_\phi(A)| > \text{ded } |A|$ for some infinite set A, $|A| = \lambda$.
(3) $|S_\phi(A)| = 2^{|A|}$ for some infinite set A, $|A| = \lambda$.

Fact 2.16. [She78, II.4.12] Let T be a complete theory in countable language, and $f_T(\lambda) := \{|S(M)| \mid M \models T, \|M\| = \lambda\}$. Then $f_T(\lambda)$ is exactly one of: λ, $\lambda + 2^{\aleph_0}$, $\lambda^{\aleph_0}$, ded λ, $(\text{ded } \lambda)^{\aleph_0}$ or 2^λ. See also [Kei76].

It is reasonable to propose the following definition:

Definition 2.17. Let K be an AEC, $\lambda \geq LS(K)$. K_λ has NIP if for all $M \in K_\lambda$, $|gS(M)| \leq \text{ded } \lambda$.

At present it is unclear that we have discovered the "correct" notion. In fact, it is plausible that there are several different notions that are equivalent when K is an elementary class, but distinct for some non-elementary K. One weakness of our definition is that unlike the corresponding first order notion, it is probably not absolute.

Grossberg raised the following question:

Question 2.18. Is there an equivalent notion which is absolute (at least for AECs K with $LS(K) = \aleph_0$ which are also $PC_{\aleph_0}$)?

Fact 2.19. [JS13, 2.5.8] Assume K has JEP, AP and NMM. Suppose there is $S^{bs} \subseteq gS$ family of types on K satisfying only (Density), (Invariance), and for all $M \in K_\lambda$, $|S^{bs}(M)| \leq \lambda^+$. See Definitions 3.1 and 3.3.

(1) If $\langle M_\alpha \in K_\lambda \mid \alpha < \lambda^+ \rangle$ is increasing and continuous, and there is a stationary set $S \subseteq \lambda^+$ such that for every $\alpha \in S$ and every model N, with $M_\alpha \leq_K N$, there is a type $p \in S^{bs}(M_\alpha)$ which is realized in M_{λ^+} and in N, then M_{λ^+} is saturated in λ^+ over λ and full over M_0.
(2) For all $M \in K_\lambda$, $|gS(M)| \leq \lambda^+$.

The following is an example of an AEC satisfying NIP that is not elementary or stable.

Example 2.20. [JS13, 2.2.4] Let λ be a cardinal. Let P be a family of λ^+ subsets of λ. Let $\tau := \{R_\alpha : \alpha < \lambda\}$ where each R_α is an unary predicate. Let K be the class of models M for τ such that for each $a \in |M|$, $\{\alpha \in \lambda \mid M \models R_\alpha(a)\} \in P$. Note that K is not elementary. Let $\leq_K$ be the substructure relation on K. The trivial λ-frame on K_λ satisfies the axioms of a semi-good λ-frame [JS13, 2.1.3], so in particular by Fact 2.19 K_λ satisfies NIP. On the other hand, it is unstable.

The next is an algebraic example of an AEC that satisfies NIP and is not elementary or stable.

Example 2.21. (ded λ = (ded $\lambda)^{\aleph_0}$) Let K be the class of real closed fields, and $F \leq_K L$ if and only if $F \preceq L$ and L/F is a normal extension, so $(K, \leq_K)$ is not elementary. Since $(K, \preceq)$ is NIP but unstable, the number of $L_{\omega,\omega}$ syntactic types over $M \in K_\lambda$, which are orbits of $\mathrm{Aut}_M(\mathfrak{C})$, coincide with Galois types $gS(M)$. The number of types is bounded by ded λ = (ded $\lambda)^{\aleph_0}$ but strictly more than λ.

Definition 2.22. [She09d, VI.2.9]

(1) For $M \in K$ and $\Gamma \subseteq gS(M)$, Γ is *inevitable* if for all $N >_K M$ there is $a \in |N| - |M|$ with $\mathbf{gtp}(a/M, N) \in \Gamma$.
(2) For $M \in K$ and $\Gamma \subseteq gS(M)$, Γ is S_**-inevitable* if for all $N >_K M$, if there is $p \in S_*(M)$ realized in N then there is $q \in \Gamma$ realized in N.

Definition 2.23. [She09d, VI.1.12(1)] We say S_* is a $\leq_{K_\lambda}$*-type-kind* when:

(1) S_* is a function with domain K_λ.
(2) $S_*(M) \subseteq gS(M)$ for all $M \in K_\lambda$.
(3) $S_*(M)$ commutes with isomorphisms.

Definition 2.24. [She09d, VI.1.12(2)] We say S_1 is *hereditarily in* S_2 when: for $M \leq_K N$ and $p \in S_2(N)$ we have $p \restriction_M \in S_1(M) \implies p \in S_1(N)$.

Definition 2.25. Let $M \in K_\lambda$. $p \in gS(M)$ is $<\mu$*-minimal* if for all $M \leq N \in K_\lambda$, $|\{q \in gS(N) : q \restriction_M = p\}| < \mu$.

$$S^{<\mu-minimal}(M) := \{p \in gS(M) \mid p \text{ is } <\mu\text{-minimal}\}.$$

Remark 2.26. $S^{<\mu-minimal}$ and $S^{\lambda-al}$ (defined in Lemma 3.14) are hereditary in gS.

The following principle known as the weak diamond was introduced by Devlin and Shelah [DS78].

Definition 2.27. Let $S \subseteq \lambda^+$ be a stationary set. $\Phi^2_{\lambda^+}(S)$ holds if and only if $\forall F : (2^\lambda)^{<\lambda^+} \to 2$ $\exists g : \lambda^+ \to 2$ such that $\forall f : \lambda^+ \to 2^\lambda$ the set $\{\alpha \in S : F(f \restriction_\alpha) = g(\alpha)\}$ is stationary.

Fact 2.28.

(1) $2^\lambda < 2^{\lambda^+}$ if and only if $\Phi^2_{\lambda^+}(\lambda^+)$ holds.
(2) $\Phi^2_{\lambda^+}(S)$ holds for a stationary set $S \subseteq \lambda^+$ if and only if $\forall F : (2 \times 2 \times \lambda^+)^{<\lambda^+} \to 2 \ \exists g : \lambda^+ \to 2$ such that $\forall \eta \in 2^{\lambda^+} \forall \nu \in 2^{\lambda^+} \forall h : \lambda^+ \to \lambda^+$ the set $\{\alpha \in S : F(\eta \restriction_\alpha, \nu \restriction_\alpha, h \restriction_\alpha) = g(\alpha)\}$ is stationary.
(3) If $\Phi^2_{\lambda^+}(\lambda^+)$ holds then there exists $\{S_i \subseteq \lambda^+ : i < \lambda^+\}$ pairwise disjoint stationary sets such that $\Phi^2_{\lambda^+}(S_i)$ for each $i < \lambda^+$.

Fact 2.29. [She09d, VI.2.18] Assume λ-AP. We have $I(\lambda^+, K) = 2^{\lambda^+}$ when:

(1) $2^\lambda < 2^{\lambda^+}$.
(2) $\mathrm{cf}(\mu) \geq \lambda^+$.
(3) $S_* \subseteq S^{<\mu-minimal}$.
(4) $|S_*(M_*)| \geq \mu$ for some $M_* \in K_\lambda$.
(5) if $M_* \leq_K M \in K_\lambda$, no subset of $S_*(M)$ of size $< \mu$ is S_*-inevitable.

Fact 2.30. [She09d, VI.2.11(2)] For every $M \in K_\lambda$ we have $|S_*(M)| \leq \lambda$ when:

(1) K has AP in λ.
(2) S_* is a hereditary $\leq_{K_\lambda}$-type-kind in gS.
(3) For every $M \in K_\lambda$ there is an S_*-inevitable $\Gamma_M \subseteq gS(M)$ of cardinality $\leq \lambda$.

3. The w*-good Frame

In this section we define w*-good frames, and show that K_λ has NIP if and only if K has a w*-good λ-grame under additional assumptions.

Definition 3.1. [She09c, III.0] Let $\lambda < \mu$, where λ is a cardinal, and μ is a cardinal or ∞. A *pre-$[\lambda,\mu)$-frame* is a triple $\mathfrak{s} = (K, \mathop{\smile}\limits^{\mid}, S^{bs})$ such that:

(1) K is an AEC with $\lambda \geq LS(K)$ and $K_\lambda \neq \emptyset$.
(2) $S^{bs} \subseteq \bigcup_{M \in K_{[\lambda,\mu)}} gS(M)$. Let $S^{bs}(M) := gS(M) \cap S^{bs}$.
(3) $\mathop{\smile}\limits^{\mid}$ is a relation on quadruples (M_0, M_1, a, N), where $M_0 \leq_K M_1 \leq N$, $a \in |N|$ and $M_0, M_1, N \in K_{[\lambda,\mu)}$. We write $a \mathop{\smile}\limits^{\mid}{}_{M_0}^{N} M_1$, or we say $\mathbf{gtp}(a/M_1, N)$ does not fork over M_0 when the relation $\mathop{\smile}\limits^{\mid}$ holds for (M_0, M_1, a, N).
(4) (Invariance) If $f : N \cong N'$ and $a \mathop{\smile}\limits^{\mid}{}_{M_0}^{N} M_1$, then $f(a) \mathop{\smile}\limits^{\mid}{}_{f[M_0]}^{N'} f[M_1]$. If $\mathbf{gtp}(a/M_0, N) \in S^{bs}(M_1)$, then $\mathbf{gtp}(f(a)/f[M_1], N') \in S^{bs}(f[M_1])$.
(5) (Monotonicity) If $a \mathop{\smile}\limits^{\mid}{}_{M_0}^{N} M_1$ and $M_0 \leq_K M_0' \leq_K M_1' \leq_K M_1 \leq_K N' \leq_K N \leq_K N''$ with $N'' \in K_{[\lambda,\mu)}$ and $a \in |N'|$, then $a \mathop{\smile}\limits^{\mid}{}_{M_0'}^{N'} M_1'$ and $a \mathop{\smile}\limits^{\mid}{}_{M_0'}^{N''} M_1'$.

(6) (Non-forking Types are Basic) If $a \underset{M}{\overset{N}{\downarrow}} M$ then $\mathbf{gtp}(a/M, N) \in S^{bs}(M)$.

Definition 3.2. [MA20, 3.6] A pre-$[\lambda, \mu)$-frame $\mathfrak{s} = (K, \downarrow, S^{bs})$ is a *w-good frame* if:

(1) $K_{[\lambda,\mu)}$ has AP, JEP and NMM.
(2) (Weak Density) For all $M <_K N \in K_\lambda$, there is $a \in |N| - |M|$ and $M' \leq N' \in K_\lambda$ such that $(a, M, N) \leq (a, M', N')$ and $\mathbf{gtp}(a/M', N') \in S^{bs}(M')$.
(3) (Existence of Non-Forking Extension) If $p \in S^{bs}(M)$ and $M \leq_K N$, then there is $q \in S^{bs}(N)$ extending p which does not fork over M.
(4) (Uniqueness) If $M \leq_K N$ both in $K_{[\lambda,\mu)}$, $p, q \in S^{bs}(N)$ both do not fork over M, and $p \restriction_M = q \restriction_M$, then $p = q$.
(5) (Continuity) If $\delta < \mu$ a limit ordinal, $\langle M_i \mid i \leq \delta \rangle$ increasing and continuous, $\langle p_i \in S^{bs}(M) \mid i < \delta \rangle$, and $i < j < \delta$ implies $p_j \restriction M_i = p_i$, and $p_\delta \in S(M_\delta)$ is an upper bound for $\langle p_i \mid i < \delta \rangle$, then $p \in S^{bs}(M_\delta)$. Moreover, if each p_i does not fork over M_0 then neither does p_δ.

Definition 3.3. A pre-$[\lambda, \mu)$-frame $\mathfrak{s} = (K, \downarrow, S^{bs})$ is a *w*-good* frame if $\mathfrak{s}$ satisfies:

(1) $K_{[\lambda,\mu)}$ has AP, JEP and NMM.
(2) (Uniqueness).
(3) (Basic NIP) For all $M \in K_{[\lambda,\mu)}$ $|S^{bs}(M)| \leq \text{ded } \|M\|$.
(4) (Few Non-Basic Types) For all $M \in K_\lambda$, $|gS(M) - S^{bs}(M)| \leq \lambda$.
(5) (Continuity$^-$) If $\delta < \mu$ a limit ordinal, $\langle M_i \mid i \leq \delta \rangle$ increasing and continuous, $\langle p_i \in S^{bs}(M_i) \mid i < \delta \rangle$, and $i < j < \delta$ implies $p_j \restriction_{M_i} = p_i$, and $p_\delta \in gS(M_\delta)$ is an upper bound for $\langle p_i \mid i < \delta \rangle$. If each p_i does not fork over M_0 then $p_\delta \in S^{bs}(M_\delta)$ and p_δ also does not fork over M_0.
(6) (Transitivity) if $p \in S^{bs}(M_2)$ does not fork over $M_1 \leq_K M_2$, and $p \restriction_{M_1}$ does not fork over $M_0 \leq_K M_1$, then p does not fork over M_0.

Remark 3.4. (Continuity-) is weaker than (Continuity). Without not forking over M_0 one cannot deduce that $p_\delta \in S^{bs}(M_\delta)$.

Remark 3.5. In a w-good frame (Transitivity) is implied by several other properties including (Existence of Non-Forking Extension). For a w*-good frame, where (Existence of Non-Forking Extension) does not hold in general, we need to explicitly include (Transitivity) as an axiom.

Definition 3.6. When $\mu = \lambda^+$ in the previous definitions, we say $\mathfrak{s}$ is a pre-/w-good/w*-good λ-frame.

From now on we build a w*-good λ-frame on K assuming the following:

Hypothesis 3.7 ($2^{\lambda^+} > 2^\lambda$)**.** We fix K an AEC and a cardinal $\lambda \geq LS(K)$ such that K_λ has AP, JEP and NMM. Assume $1 \leq I(\lambda^+, K) < 2^{\lambda^+}$, and K_λ has NIP.

If K is categorical in λ, then K has λ-AP by the following fact, which appeared in [She87, 3.5] first, and a clearer proof can be found in [Gro02, 4.3]. λ-JEP follows from categoricity, and λ-NMM follows from categoricity and $K_{\lambda^+} \neq \emptyset$.

Fact 3.8. [She87, 3.5] ($2^\lambda < 2^{\lambda^+}$) If $I(\lambda, K) = 1 \leq I(\lambda^+, K) < 2^{\lambda^+}$, then K has the λ-AP.

Thus we could also assume:

Hypothesis 3.9. We fix K an AEC and a cardinal $\lambda \geq LS(K)$ such that K is categorical in λ. Assume $2^{\lambda^+} > 2^\lambda$, $1 \leq I(\lambda^+, K) < 2^{\lambda^+}$, and K_λ has NIP.

Definition 3.10. $p = \mathbf{gtp}(a/M, N)$ has the extension property if for all K-embedding $f : M \to M_1 \in K_\lambda$ there is $q \in gS(M_1)$ extending $f(p)$.

Definition 3.11. $p = \mathbf{gtp}(a/M, N)$ is λ-*unique* if

(1) $p = \mathbf{gtp}(a/M, N)$ has the extension property.
(2) $\mathbf{gtp}(a, M, N) \leq_{h_l} \mathbf{gtp}(a_l, M', N_l)$, and $\mathbf{gtp}(a_l/M', N_l)$ have the extension property, for $l = 1, 2$, then $\mathbf{gtp}(a_1/M', N_1) = \mathbf{gtp}(a_2/M', N_2)$.

Fact 3.12. [She09d, VI.2.5(2B)] If K_λ has AP and $\lambda \geq LS(K)$, $\mathbf{gtp}(a, M, N)$ has $\geq \lambda^+$ realizations in some extension of M (necessarily in $K_{\geq\lambda^+}$) if and only if $\mathbf{gtp}(a/M, N)$ has the extension property.

Now we define the w*-good λ-frame.

Definition 3.13. The preframe $\mathfrak{s}_{\lambda-unq}$ is defined such that:

(1) $S^{bs}(M) := \{p = \mathbf{gtp}(a/M, N) \mid p \text{ has the extension property}\}$.
(2) $p = \mathbf{gtp}(a/M, N) \in S^{bs}(M)$ does not fork over $M_0 \leq_K M$ if $p \restriction_{M_0}$ is λ-unique.

Lemma 3.14. $S^{\lambda-al}(M) := \{p \in gS(M) \mid p \text{ has } \leq \lambda\text{-many realizations}\}$ satisfies $|S^{\lambda-al}(M)| \leq \lambda$. By realizations we mean realizations in any $\leq_K$-extension of M in K_{λ^+}. So $\mathfrak{s}_{\lambda-unq}$ satisfies (Few Non-Basic Types).

Proof. Suppose not, i.e. $|S^{\lambda-al}(M)| \geq \lambda^+$.
Claim: There is no $\Gamma \subseteq S^{\lambda-al}(M)$, $|\Gamma| \leq \lambda$ that is inevitable.

Otherwise, suppose there exists such Γ. By Fact 2.30, taking S_* to be gS, and Γ_M to be Γ, we have $|gS(M)| \leq \lambda$, so in particular $|S^{\lambda-al}(M)| \leq \lambda$, contradiction.

Now by the claim and Fact 2.29, taking S_* there to be $S^{\lambda-al}$ and μ there to be λ^+, we have $I(\lambda^+, K) = 2^{\lambda^+}$, contradiction. ☐

Thus from now on in this section we also assume $|S^{\lambda-al}(M)| \leq \lambda$.

Lemma 3.15. $\mathfrak{s}_{\lambda-unq}$ satisfies the following properties in Definitions 3.1, 3.2 and 3.3:

(1) (Invariance).
(2) (Monotonicity).
(3) (Non-Forking Types are Basic).
(4) AP, JEP and NMM.
(5) (Basic NIP).
(6) (Uniqueness).
(7) (Transitivity).

Proof. Easy. We prove (Transitivity) as an example. Suppose $p \in S^{bs}(N)$ does not fork over $M_1 \leq_K N$, and $p \restriction_{M_1}$ does not fork over $M_0 \leq_K M_1$. Then $(p \restriction_{M_1}) \restriction_{M_0}$ is λ-unique, i.e. $p \restriction_{M_0}$ is. Thus p does not fork over M_0. $\square$

Lemma 3.16 (ded $\lambda = \lambda^+ < 2^\lambda$)**.** Suppose that $\mathfrak{s}_{\lambda-unq}$ satisfies (Continuity). If $p \in S^{bs}(M)$, then there is $N \geq_K M$ and $q \in S^{bs}(N)$ extending p that does not fork over N. In particular, for any $N' \geq_K N$ there is unique $q' \in gS(N')$ extending q that does not fork over N.

Proof. It suffices to show that there is a λ-unique type above any basic type. By Fact 2.19 let $\mathfrak{C} \in K_{\lambda^+}$ be saturated in λ^+ over λ. It is also homogeneous in λ^+ over λ by Fact 2.13. Let $(a, M, N) \in K^3_\lambda$ such that $\mathbf{gtp}(a/M, N)$ has the extension property and there is no λ-unique type above $\mathbf{gtp}(a/M, N)$. Build $(a_\eta, M_\eta, N_\eta) \in K^3_\lambda$ for $\eta \in^{<\lambda} 2$ and $h_{\eta,\nu}$ for $\eta < \nu \in^{<\lambda} 2$ such that:

(1) $(a_{\langle\rangle}, M_{\langle\rangle}, N_{\langle\rangle}) = (a, M, N)$.
(2) $(a_\eta, M_\eta, N_\eta) \leq_{h_{\eta,\nu}} (a_\nu, M_\nu, N_\nu)$ for $\eta < \nu$.
(3) $h_{\eta,\rho} = h_{\nu,\rho} \circ h_{\eta,\nu}$ for $\eta < \nu < \rho$.
(4) $M_{\eta^\frown 0} = M_{\eta^\frown 1}$, $N_{\eta^\frown 0} = N_{\eta^\frown 1}$, and $h_{\eta,\eta^\frown 0} \restriction M_\eta = h_{\eta,\eta^\frown 1} \restriction M_\eta$.
(5) $\mathbf{gtp}(a_{\eta^\frown 0}, M_{\eta^\frown 0}, N_{\eta^\frown 0}) \neq \mathbf{gtp}(a_{\eta^\frown 1}, M_{\eta^\frown 1}, N_{\eta^\frown 1})$, both having λ^+-many realizations.
(6) If $\eta \in^\delta 2$ for δ a limit ordinal, take M_η and N_η to be directed colimits.

Construction: Base case and limit case are clear. At successor stage use non-λ-uniqueness to get two distinct extensions, each having λ^+-many realizations.
Enough: Let $M \leq_K \mathfrak{C} \in K_{\lambda^+}$ be saturated over λ. Build $g_\eta : M_\eta \to \mathfrak{C}$ for $\eta \in^\lambda 2$ such that:

(1) $g_\nu \circ h_{\eta,\nu} = g_\eta$ for $\nu < \eta$.
(2) $g_{\eta^\frown 0} = g_{\eta^\frown 1}$

This is possible: Base case take $g_{\langle\rangle}$ to be inclusion $M \leq_K \mathfrak{C}$. At limit use the universal property of M_η as a directed colimit. At $\alpha = \beta + 1$, suppose we have g_η.

(1)
$$\begin{array}{ccccccc}
\mathfrak{C} & \xleftarrow[j]{} & M''_{\eta^\frown 0} & \xleftarrow[\cong_h]{} & M'_{\eta^\frown 0} & \xrightarrow{\cong_g} & M_{\eta^\frown 0} \\
{\scriptstyle id}\nwarrow & & {\scriptstyle id}\uparrow & & {\scriptstyle id}\uparrow & & {\scriptstyle id}\uparrow \\
 & & g_\eta[M_\eta] & \xleftarrow[\cong_{g_\eta}]{} & M_\eta & \xrightarrow{\cong h_{\eta,\eta^\frown 0}} & h_{\eta,\eta^\frown 0}[M_\eta]
\end{array}$$

Use basic extension to obtain the right square and g, and then obtain the middle square and h. Finally the left triangle is by saturation of $\mathfrak{C}$. Now define $g_{\eta^\frown 0} = g_{\eta^\frown 1}$ to be the composition of the top row from right to left.

This is enough: For each branch $\eta \in^\lambda 2$, take directed colimit to obtain (a_η, M_η, N_η). Obtain $f_\eta : M_\eta \to \mathfrak{C}$ by the universal property of colimits such that $f_\eta \circ h_{\nu,\eta} = g_\nu$ for all $\nu < \eta$, and obtain $f'_\eta : N_\eta \to \mathfrak{C}$ extending f_η by saturation over λ. Since each $f'_\eta(a) \in |\mathfrak{C}|$, but $\|\mathfrak{C}\| = \operatorname{ded} \lambda < 2^\lambda$, there must be $\eta, \nu \in^\lambda 2$ such that $f'_\eta(a) = f'_\nu(a)$. Let $\alpha < \lambda$ be the least such that $\eta(\alpha) \neq \nu(\alpha)$. Without loss of generality say $\eta(\alpha) = 0$ and $\nu(\alpha) = 1$. Then the following diagram commutes:

(2)
$$\begin{array}{ccc}
N_{\eta\restriction_\alpha{}^\frown 0} & \xrightarrow{f'_\eta \circ h_{\eta\restriction_\alpha{}^\frown 0,\eta}} & \mathfrak{C} \\
{\scriptstyle id}\uparrow & f'_\nu \circ h_{\eta\restriction_\alpha{}^\frown 1,\nu} & \uparrow \\
M_{\eta\restriction_\alpha{}^\frown 0} & \xrightarrow{id} & N_{\eta\restriction_\alpha{}^\frown 1}
\end{array}$$

with $f'_\eta \circ h_{\eta\restriction_\alpha{}^\frown 0,\eta}(a_{\eta\restriction_\alpha{}^\frown 0}) = f'_\nu \circ h_{\eta\restriction_\alpha{}^\frown 1,\nu}(a_{\eta\restriction_\alpha{}^\frown 1})$ since $f'_\eta(a_\eta) = f'_\nu(a_\nu)$, contradicting requirement (5) of the construction. □

Remark 3.17. The proof of Lemma 3.16 is along the argument of Mazari-Armida in [MA20, 4.13] and [She09d, VI.2.25], and the difference is that there the saturated model over λ lies in $K_{\lambda^{++}}$. For completeness we included all the details.

Question 3.18. Lemma 3.16 is a weaker form of (Existence of Non-Forking Extension). Is it possible to obtain (Existence of Non-Forking Extension) in its full strength, by perhaps considering another family of basic types and non-forking relation? One could imitate the w-good λ-frame in [MA20] and use λ-unique types as basic ones, and then Lemma 3.16 gives a proof of (Weak Density). However, then we it is hard to show that having such a frame implies NIP.

The following definition is [She99, 1.8], which is defined for types of any finite length. Here we only need it for length 1. Thus we use the version from [Bal09, 11.4(1)].

Definition 3.19. (1) K is (κ, λ)-*local* if for every increasing continuous chain $M = \bigcup_{i<\kappa} M_i$ with $\|M\| = \lambda$ and for any $p, q \in gS(M)$: if $p \restriction_{M_i} = q \restriction_{M_i}$ for all i then $p = q$.

(2) K is $(<\kappa,\lambda)$-*local* if K is (μ,λ)-local for all $\mu<\kappa$.

Lemma 3.20. If K is $(<\lambda^+,\lambda)$-local, then $\mathfrak{s}_{\lambda-unq}$ has (Continuity$^-$).

Proof. Let M_i, $i<\delta$ be increasing continuous. $p_i\in S^{bs}(M_i)$ increasing and for $i<j<\delta$ we have $p_j\restriction_{M_i}=p_i$, and p_δ upper bound. Suppose p_δ has $\leq\lambda$-many realizations. Then there is a set S of cardinality λ^+ of realizations of p_0, such that for each $a\in S$, by locality there is $i<\delta$ such that a realizes p_i but not p_{i+1}. By pigeonhole principle for some $i<\delta$ there are λ^+-many realizations of p_i that are not realizations of p_{i+1}. Since there are $\leq\lambda$-many types in $S(M_{i+1})$ that have $\leq\lambda$-many realizations, there must be another type in $S(M_{i+1})$ with λ^+ realizations distinct from p_{i+1}, which contradicts λ-uniqueness of p_{i+1}.

For the moreover part, if p_0 does not fork over M_0, so $p_0=p_\delta\restriction_{M_0}$ is λ-unique, i.e. p_δ does not fork over M_0. □

Theorem 3.21 $(2^{\lambda^+}>2^\lambda)$**.** Let K be an AEC with $\lambda\geq LS(K)$ with λ-AP, λ-JEP and λ-NMM, and $1\leq I(\lambda^+,K)<2^{\lambda^+}$. K_λ has NIP if and only if there is a w*-good λ-frame on K except possibly without (Continuity$^-$). Moreover,

(1) (ded $\lambda=\lambda^+<2^\lambda$) If $\mathfrak{s}_{\lambda-unq}$ satisfies in addition (Continuity), then the w*-good frame satisfies in addition that if $p\in S^{bs}(M)$, then there is $N\geq_K M$ and $q\in S^{bs}(N)$ extending p that does not fork over N. In particular, for any $N'\geq_K N$ there is $q'\in gS(N')$ extending q that does not fork over N.
(2) if K is $(<\lambda^+,\lambda)$-local, then the frame has (Continuity$^-$).

Proof. The moreover part follows from Lemma 3.16. □

Corollary 3.22 $(2^{\lambda^+}>2^\lambda)$**.** Let K be an AEC categorical in $\lambda\geq LS(K)$, and $1\leq I(\lambda^+,K)<2^{\lambda^+}$. K_λ has NIP if and only if there is a w*-good λ-frame on K except possibly without (Continuity$^-$). Moreover,

(1) (ded $\lambda=\lambda^+<2^\lambda$) If $\mathfrak{s}_{\lambda-unq}$ satisfies in addition (Continuity), then the w*-good frame satisfies in addition that if $p\in S^{bs}(M)$, then there is $N\geq_K M$ and $q\in S^{bs}(N)$ extending p that does not fork over N. In particular, for any $N'\geq_K N$ there is $q'\in gS(N')$ extending q that does not fork over N.
(2) if K is $(<\lambda^+,\lambda)$-local, then $\mathfrak{s}_{\lambda-unq}$ has (Continuity$^-$).

4. Syntactic independence property

In this section we assume tameness, and use Galois Morleyization to show that the negation of NIP leads to being able to encode subsets, as a parallel of first order independence property.

Hypothesis 4.1. Let κ be an infinite cardinal and K an AEC. Let $\tau=L(K)$ be its underlying language.

We first extend the definition of Galois types to longer lengths and set-valued domains.

Definition 4.2. (1) $K^3 := \{(\bar{a}, A, N) \mid N \in K, A \subseteq |N|, \bar{a}$ is a sequence from $|N|\}$.
(2) For $(\bar{a}_0, A, N_0), (\bar{a}_1, A, N_1) \in K^3$, $(a_0, A, N_0)E_{at}(a_1, A, N_1)$ if there are $N \in K$, $f_0 : N_0 \to_A N$, and $f_1 : N_1 \to_A N$ K-embeddings such that $f_0(\bar{a}_0) = f_1(\bar{a}_1)$, $f_0 \restriction_A = f_1 \restriction_A$.
(3) E is the transitive closure of E_{at}.
(4) For $(\bar{a}, A, N) \in K^3$, *the Galois type of* $\bar{a}$ *over* A *in* N is $\mathbf{gtp}(a/A, N) := [(a, A, N)]_E$.
(5) For $N \in K$ and $A \subseteq |N|$, α an ordinal or ∞, $gS^{<\alpha}(A; N) := \{\mathbf{gtp}(\bar{a}/A, N) \mid (\bar{a}, A, N) \in K^3$ and $\bar{a} \in^{<\alpha} |N|\}$. $gS^{\alpha}(A; N)$ is defined similarly.

Remark 4.3. In the case where $A = |M|$ for $M \in K$, $\bigcup_{N \geq_K M} gS^1(|M|, N)$ is what we defined as $gS(M)$ in Definition 2.5.

The following technique first appeared in [Vas16c], which allows one to work with Galois types in a syntactic way.

Definition 4.4. Let κ be an infinite cardinal and K an AEC. The $(<\kappa)$-*Galois Morleyization* of K is $\hat{K}$, an AEC in a $(<\kappa)$-ary language $\hat{\tau}$ extending τ such that:

(1) The structures and the substructure relation $\leq_{\hat{K}}$ in $\hat{K}$ are the same as K.
(2) For each $p \in gS^{<\kappa}(\emptyset)$, there is a predicate of the same length $R_p \in \hat{\tau}$. For each $M \in K$ and $\bar{p}$, define $M \models R_p[\bar{a}]$ if and only if $\mathbf{gtp}(\bar{a}/\emptyset, M) = p$. By extension, one can interpret quantifier-free $L_{\kappa,\kappa}(\hat{\tau})$ formulas.
(3) The $(<\kappa)$-syntactic type of $\bar{a} \in^{<\kappa} |M|$ over $A \subseteq |M|$ is $\mathbf{tp}_{\text{qf-}L_{\kappa,\kappa}(\hat{\tau})}(\bar{a}/A, M)$, the set of all quantifier-free $L_{\kappa,\kappa}(\hat{\tau})$ formulas with parameters from A that $\bar{a}$ satisfies. For a particular quantifier-free $L_{\kappa,\kappa}(\hat{\tau})$-formula $\phi(\bar{x}, \bar{y})$, $\mathbf{tp}_{\phi}(\bar{b}/A, M) := \{\phi(\bar{x}, \bar{a}) \mid \bar{a} \in A, M \models \phi(\bar{b}, \bar{a})\}$.
(4) For $M \in K$ and $A \subseteq |M|$, $S^{<\alpha}_{\text{qf}-L_{\kappa,\kappa}(\hat{\tau})}(A; M) := \{\mathbf{tp}(\bar{b}/A, M) \mid \bar{b} \in^{<\alpha} |M|\}$

Remark 4.5. There are $\leq 2^{<(LS(K)^+ + \kappa)}$ formulas in $\hat{\tau}$.

Fact 4.6. [Vas16c, 3.18(2)] Under the notation of the previous definition, for each ordinal α, $M \in K$, $A \subseteq M$, $\mathbf{gtp}(\bar{b}/A, M) \mapsto \mathbf{tp}_{\text{qf-}L_{\kappa,\kappa}(\hat{\tau})}(\bar{b}/A, M)$ from $gS^{\alpha}(A; M)$ to $S^{\alpha}_{\text{qf}-L_{\kappa,\kappa}(\hat{\tau})}(A; M)$ is bijective if and only if K is $(<\kappa)$-tame.

Fact 4.7. [Gro, 2.7.29] (Morley's method) Let T be a first order theory with built-in Skolem functions and Γ a set of T-types. Let p_n be a T-type in n variables and c_n a new constant for each $n < \omega$ such that:

(1) $T^* \supseteq T \cup \{p_n(c_0, \ldots, c_n) \mid n < \omega\}$ is consistent
(2) Each p_n is realized in some $M \in EC(T, \Gamma)$.

Then there is $N \in EC(T^*, \Gamma)$.

Theorem 4.8. Suppose K is $(< \aleph_0)$-tame, $M \in K$, $C \subseteq |M|$, $\lambda := \|C\| \geq \beth_3(LS(K))$ and $(\text{ded } \lambda)^{2^{LS(K)}} = \text{ded } \lambda$. Suppose $|gS^1(C; M)| > \text{ded } \lambda$. Then there is $N \in K$, $\langle \bar{a}_n \in^m |N| \mid n < \omega \rangle$ and ϕ in the language of Galois Morleyization such that for every $n < \omega$ and $w \subseteq n$ there is $b_w \in |N|$ such that for all $i < n$,

$$N \models \phi(\bar{a}_i, b_w) \iff i \in w$$

Proof. Let $\hat{K}$ be the $(< \aleph_0)$ Galois Morleyization of K. Note that both classes have the same Galois types. By Shelah's Presentation Theorem $\hat{K} = PC(T, \Gamma, \hat{\tau})$ with $|T| \leq 2^{LS(K)}$, with the language of T containing $\hat{\tau}$. Then by tameness and the previous fact $|S^1_{\text{qf}-L_{\omega,\omega}(\hat{\tau})}(C; M)| > \text{ded } \lambda$, so for some quantifier-free formula $\phi(\bar{y}, x)$ in $L_{\omega,\omega}(\hat{\tau})$ with $|S_\phi(C; M)| > \text{ded } \lambda$, since there are $\leq 2^{LS(K)}$-many quantifier-free $L_{\omega,\omega}(\hat{\tau})$-formulas.

Without loss of generality $C = \lambda = |C|$. Let $\mu := (\text{ded } \lambda)^+$. For notational simplicity we view $S_\phi(C; M)$ as S, a family of subsets of ${}^{\ell(\bar{y})}C$, where

$$A \in S \iff \{\phi(\bar{a}, x) \mid \bar{a} \in A\} \in S_\phi(C).$$

We also assume $\bar{y}$ has length 1. The proof for other cases is similar.

Claim: For all $\alpha < \lambda$, if $|\{A \cap \alpha \mid A \in S\}| \geq \mu$, then $\alpha \geq (\beth_2(LS(K)))^+$.
Proof of Claim: Suppose there is $\alpha < \lambda$, $|\{A \cap \alpha \mid A \in S\}| \geq \mu$. Since $\{A \cap \alpha \mid A \in S\}$ is the set of branches of the a subtree of ${}^{<\alpha}2$, $\text{ded } \lambda < \mu \leq \text{ded } |{}^{<\alpha}2| \leq \text{ded } 2^{|\alpha|}$, so $2^{|\alpha|} > \lambda \geq \beth_3(LS(K))$, so $|\alpha| > \beth_2(LS(K))$. Thus the claim holds.

We may assume $\lambda > \beth_2(LS(K))$ and for all $\alpha < \lambda$, $|\{A \cap \alpha \mid A \in S\}| < \mu$. If this holds, then we are done since $\lambda \geq \beth_3(LS(K)) > \beth_2(LS(K))$. If not, replace λ with smallest $\alpha < \lambda$ such that $|\{A \cap \alpha \mid A \in S\}| \geq \mu$. By minimality for all $\beta < \alpha$, $|\{A \cap \beta \mid A \in S\}| < \mu$. Such α might be small, but by the claim $\alpha \geq (\beth_2(LS(K)))^+$, and this is enough for the arguments of the rest of the argument.

For each $\alpha \leq \lambda$ let $S^0_\alpha := \{\langle A \cap \alpha, \alpha \rangle \mid A \in S\}$. $\bigcup_\alpha S^0_\alpha$ is a tree when equipped with

$$(A_1, \alpha_1) \leq (A_2, \alpha_2) \iff \alpha_1 \leq \alpha_2 \wedge A_1 = A_2 \cap \alpha_1.$$

Let

$$S^1_\alpha := \{s \in S^0_\alpha \mid |\{t \in S^0_\alpha \mid s \leq t\}| \geq \mu\},$$

and

$$S^1_\lambda := \{s \in S^0_\lambda \mid \forall \alpha < \lambda (s \restriction_\alpha \in S^1_\alpha)\}.$$

We build $S_n \subseteq S^1_\lambda$ for $n < \omega$, $\lambda > \alpha^A_i(n, 0) > \ldots > \alpha^A_i(n, n-1) > i$ for each $i \in S_n$ and $(A, i) \in S^1_i$, and $p_n \in S^n_T(\emptyset)$ such that:

(1) $S_0 = S^1_\lambda$.
(2) $|S_n| = \lambda$ for all n

(3) $S_{n+1} \subseteq S_n$ for all n.
(4) $p_n \subseteq p_{n+1}$ for all n.
(5) For all $n < m$, $(A,i) \in S_n$ and $(B,j) \in S_m$, $(A,i) \leq (B,j) \in \bigcup_\alpha S^0_\alpha$

$$p_n = \mathbf{tp}_T(\langle \alpha^A_i(n,0), \dots \alpha^A_i(n,n-1)\rangle/\emptyset, M) = \mathbf{tp}_T(\langle \alpha^B_j(m,0), \dots \alpha^B_j(m,n-1)\rangle\emptyset, M).$$

(6) For all $(A,i) \in S_n$ and $w \subseteq n$ there is $(A_w,\lambda) \in S^1_\lambda$ such that $(A,i) \leq (A_w,\lambda)$ and $\alpha^A_i(n,i) \in A_w \iff i \in w$.

Construction: We build these objects by induction on n. When $n = 0$ there is nothing to do. Assume we have built S_n, $\alpha^A_i(n,j)$ for $(A,i) \in S_n$ and p_n.

Fix $s = (A,i) \in S_n$. Clearly $T_s := \{t \in \bigcup_{\beta<\lambda} S^1_\beta \mid s \leq t\}$ is a tree. For every $s \leq t \in S^1_\lambda$, $B_t := \{t^* \mid s \leq t^* \leq t\}$ is a branch of T_s, and $t_1 \neq t_2 \implies B_{t_1} \neq B_{t_2}$. Since

$$|S^0_\lambda - S^1_\lambda| = |\bigcup_{\alpha<\lambda, s\in S^0_\alpha - S^1_\alpha} \{t \in S^0_\lambda \mid s \leq t\}| < \mu,$$

T_s has $\geq \mu$-many branches, and hence $|T_s| > \lambda$. Then for some i', $|T_s \cap S^1_{i'}| > \lambda$. Let $s_j = (A_j, i') \in T_s \cap S^1_{i'}$ for $j < \lambda^+$. Now let $\alpha^A_i(n+1,k) := \alpha^{A_j}_{i'}(n,k)$ for all $k < n$. Let $\alpha^A_i(n+1,n)$ be the least α such that $s_0(\alpha) \neq s_1(\alpha)$, i.e. $\alpha \in A_0 - A_1$ or $\alpha \in A_1 - A_0$. Note that $i < \alpha^A_i(n+1,n) < i' < \alpha^A_i(n+1,n) < \ldots < \alpha_i(n+1,0)$. Since $|S_n| = \lambda \geq (\beth_2(LS(K)))^+$, and there are $\leq \beth_2(LS(K))$ T-types, by the pigeonhole there is $S_{n+1} \subseteq S_n$, $|S_{n+1}| = \lambda$ such that for all (A,i), $(B,j) \in S_{n+1}$,

$$\mathbf{tp}_T(\langle \alpha^A_i(n,0), \dots \alpha^A_i(n,n-1)\rangle/\emptyset, M) = \mathbf{tp}_T(\langle \alpha^B_j(n,0), \dots \alpha^B_j(n,n-1)\rangle\emptyset, M),$$

and define this type to be p_{n+1}.

$$T^* := T \cup \{\exists x(\bigwedge_{i<n} \phi(c_i,x)^{i\in w}) \mid w \subseteq n < \omega\} \cup \{p_n(c_0,\dots,c_{n-1}) \mid n < \omega\}$$

is consistent, and by Morley's method we are done. ☐

Question 4.9. For K an elementary classes, the conclusion of the previous theorem implies that K can encode arbitrary subsets of any set by the compactness theorem. Here one can only encode subsets of $n \in \omega$.

(1) Can K encode larger subsets?
(2) Is there a Hanf number? I.e. Are there κ, μ such that if K can encode subsets of μ with size $< \kappa$, then K can encode all subsets? Grossberg conjectured that the Hanf number is $\beth_{2^{LS(K)}}$ should it exist.

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