Revisiting Zilber's Trichotomy

Jinhe Ye

Nanjing University

(Joint work with Ben Castle and Assaf Hasson)

Delta 24, Inner Mongolia University

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Throughout, T is a countable, complete first-order theory with infinite models, and definable means definable with parameters.

Definition

Let κ be some cardinal. We say T is *κ-categorical* if T has only one model of cardinality κ up to isomorphism, and T is *uncountably categorical* if it is κ-categorical for some uncountable κ.

Theorem (Morley 1965)

T is uncountably categorical iff T is κ-categorical for all $\kappa > \aleph_0$.

This theorem started modern model theory.

We work in a monster model of T.

Definition

Let X be a definable set, we say X is *strongly minimal* if every definable subset of X is either finite or cofinite.

Morley's theorem has a more conceptual description.

Theorem (Baldwin-Lachlan 1971)

If T is κ-categorical for some uncountable κ, then T has a prime model M and a strongly minimal set X is definable over M such that any $M_1, M_2 \models T$ are isomorphic if and only if $X(M_1)$ and $X(M_2)$ have the same "dimension".

Strongly Minimal Theories

The above theorem states that T is "controlled" by a strongly minimal theory T' given by the induced structure on X. We wish to classify strongly minimal theories.

Typical examples of strongly minimal theories are the following:

- Infinite sets in the empty language.
- Vector space V over a skew field K where the scalar multiplications are given as function symbols.
- Algebraically closed fields in the language of rings.

Pregeometries

Definition

A *pregeometry* is a pair (X, cl) where $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ such that:

- $\mathrm{cl}(A) \supseteq A$ for all A;
- $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ for all A;
- $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$ for all $A \subseteq B$;
- $\mathrm{cl}(A) = \bigcup\{\mathrm{cl}(A_0) \colon A_0 \subseteq A,\ A_0 \text{ finite}\}$.
- Exchange: If $a \in \mathrm{cl}(Ab) \setminus \mathrm{cl}(A)$, then $b \in \mathrm{cl}(Aa)$.

We work in a saturated model of a strongly minimal theory T and acl denotes the algebraic closure in T

- In a strongly minimal theory, acl satisfies exchange: $a \in \mathrm{acl}(Ab) \setminus \mathrm{acl}(A) \Rightarrow b \in \mathrm{acl}(Aa)$.
- We say that T is *disintegrated* if $\mathrm{acl}(A) = \bigcup_{a \in A} \mathrm{acl}(a)$.
- T *locally modular* if it is either disintegrated or the pregeometry of a linear space. The latter is also referred to as non-trivial locally modular.

Zilber's Non-Finite Axiomatizability

- Zilber: Totally categorical theories cannot be finitely axiomatized.
- To be totally categorical, any model must be of infinite "dimension".
- Key insight of Zilber's proof: The strongly minimal set arising in a totally categorical theory must be locally modular. More precisely, they essentially are infinite sets or vector spaces over finite fields, and the dimension is the usual dimension in each setting.

Zilber's Weak Trichotomy Theorem

Theorem (Zilber 1977?)

If X is strongly minimal, then one of the following holds:

- *(Trivial) X is trivial in the sense that* acl *is disintegrated.*
- *(Non-trivial locally modular) X is essentially a vector space. More precisely, possibly after adding some constant symbols to the language of X, there is an infinite abelian group G bi-interpretable with X for which every definable subset of any Cartesian power of G is a finite Boolean combination of cosets of definable subgroups.*
- *(Non-locally modular) There is a definable pseudoplane interpretable in X.*

Conjecture (Zilber's Trichotomy Conjecture)

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- *(Non-locally modular) X is bi-interpretable with an algebraically closed field.*

Theorem (Hrushovski 1993)

There are non-locally modular strongly minimal sets that don't interpret a group.

Zariski Geometries

Theorem (Hrushovski-Zilber, JAMS 1996)

The Trichotomy Conjecture holds in Zariski geometries.

- Zariski Geometries are axiomatic topological framework axiomatizing the Zariski topology on algebraic curves (and their powers) over algebraically closed fields.
- Using the Trichotomy in Zariski Geometries, we know that it holds in an abundant number of natural examples.

Abundance of Examples

- If D definable in DCF_0 and strongly minimal, then D satisfies Zilber's Trichotomy.
- Appropriate versions of trichotomy holds for thin types in seperably closed fields.
- Appropriate versions of trichotomy holds for minimal types in ACFA.

The first two bullets are the key ingredient in Hrushovski's proof of Mordell-Lang in characteristic 0 and p respectively.
The last two bullets are not strongly minimal, the last one isn't even stable.

Theorem (Rabinovich, unpublished)

The conjecture holds for reducts of ACF whose underlying set is the field.

Definition

Let M be a first-order structure. An *M-relic* is another structure (in a different language) N whose universe is definable in M^{eq} and whose interpretations of symbols are M-definable/interpretable. We say a M-relic is *strongly minimal* if its theory is strongly minimal.

The Rabinovich theorem is a special case of the Restricted Trichotomy Conjecture of Zilber.

Conjecture (Restricted Trichotomy, Proceedings of the ICM 1983)

Trichotomy holds in any strongly minimal relics of ACF.

Restricted Trichotomy

Zilber's Philosophy/Principle: Appropriate versions of the trichotomy holds in geometric structures where every there is a "tame topology" behind the definable sets.

Conjecture (Peterzil 2005)

Let M be o-minimal, any M-relic that is geometric structure satisfies an appropriate version of the trichotomy conjecture.

Conjecture (Kowalski-Randriambololona 2014)

A strongly minimal relic of ACVF satisfies Zilber's Trichotomy.

This Talk

- We will introduce an axiomatic framework for tackling Zilber's Restricted Trichotomy.
- Using this framework, we resolve Zilber's Restricted Trichotomy Conjecture in ACF by passing to ACVF.
- For the Kowalski-Randriambololona conjecture, we reduce the conjecture to a techinical question, which we verify in residue characteristic 0.

History of Restricted Trichotomy

Prior to Rabinovich's theorem, it was known for some special reducts.

Theorem (Marker-Pillay 1990)

Restricted trichotomy holds for reducts of the form $(\mathbb{C}, +, X)$*.*

Rabinovich's theorem was generalized by Hasson and Sustretov.

Theorem (Hasson-Sustretov 2017)

Let M *be a* ACF*-relic whose underlying set is an algebraic curve, then* M *satisfies the Restricted Trichotomy.*

The Restricted Trichotomy Conjecture: **Curves are the only non-locally modular strongly minimal relics. There shouldn't be any higher-dimensional non-locally modular strongly minimal relics.**

Existing Strategy for Dimension 1 Relics

- Typically sufficient to interpret a strongly minimal group.
- Non-local modularity $\rightarrow$ Rank 2 family of plane curves with good properties.
- The main idea: Show that M can 'define' when two plane curves are tangent at a diagonal point $(x, x) \in M^2$ in a weak sense.
- Consider a strongly minimal family C of curves through a generic diagonal point. One shows that tangency of a curve in C to another curve in C at the point given point is definable. The equivalence classes are the slopes and one can define a composition of curves that gives rise to operation on slopes.
- In our terminology, near a generic point, plane curves are "homeomorphisms", so one further shows that slopes can be composed generically.
- Use group configuration to construct a group from the relation given by composition of slopes.

Major Progress

Theorem (Castle, JAMS 2024)

Restricted Trichotomy holds for ACF_0-relics.

- Multiple intersections is local. It could be recovered via (Euclidean) topological information.
- In ACF_0, tangency and multiple intersections are the same.
- Non-locally modular ACF_0-relics recovers sufficient information of the Euclidean topology to detect tangency.

Question

Can we prove restricted trichotomy for ACF_p via going to $\mathrm{ACVF}_{(p,p)}$?

Axiomatic Framework for Restricted Trichotomy

- Zilber's Principle: Trichotomy is true if the underlying structure can be equipped with a "tame topology", even if the relic has no access to it on the face of it.
- We will give an axiomatic topological framework that incorporates the strategies for known cases of the trichotomy. Establish the trichotomy in this framework and verify that various structures can be fitted into this framework.

Theorem (Castle-Hasson-Y. 2024)

Let K be an algebraically closed field. Any strongly minimal K-relic is either locally modular or interprets a field definably isomorphic to K.

Definition

Let T be a first-order complete, we say T is geometric if:

- T eliminates $\exists^\infty$;
- acl satisfies exchange.

Example

- Any strongly minimal theory. E.g. ACF;
- Real closed fields, or any *o*-minimal expansions of RCF;
- ACVF;
- p-adically closed fields.

Definition

An $\aleph_1$-saturated geometric structure K is a *Hausdorff Geometirc Structure* if:

- K has a Hausdorff topology τ (that need not be definable).
- If $X \subseteq K^n$ is definable over A and $a \in \mathrm{Fr}_\tau(X)$ then $\dim(a/A) \leq \dim(X)$.
- Density of generics: Let $X \subseteq K^n$ be definable over a countable set A, and let $a \in X$ be generic over A. Let $B \supseteq A$ be countable. Then every τ-neighborhood of a contains a generic of X over B.
- Finite correspondences are homeomorphisms generically: For X, Y, and $Z \subseteq X \times Y$ are definable over A, and both of $Z \to X$ and $Z \to Y$ are finite-to-one. Let (x, y) be generic in Z over A. Then there are open neighborhoods U of x in X, V of y in Y such that $Z \cap U \times V$ is the graph of a homeomorphism $U \to V$.

HGS with Enough Open Maps

- Enough Open Maps is a technical assumption that essentially guarantees that transversal intersections are stable under local perturbation. Transverse intersections are can be described as certain maps into the parameter space are open generically.
- Using this, one could do the following: Let $X \subseteq K^2$ be a plane curve with a frontier point at (a, b). If X happens to be definable in a strongly minimal relic M, we want to "recognize" the frontier point (a, b) within M.
- Previous Work (Castle, Hasson, Sustretov, etc): This is sufficient to "define tangency" and compositions.
- If the structure admits an abstract notion of smoothness that is well-behaved (e.g. has some version of inverse function theorem and Sard's theorem attached to it), then it has enough open maps automatically.

HGS with Enough Open Maps

Example

- Complex numbers with the Euclidean topology;
- *O*-minimal expansions of RCF;
- 1-h-minimal fields;
- Topological éz fields. E.g. Henselian valued fields of characteristic 0, ACVF.

Ramification Purity

- Over the complex numbers, the ramification locus of quasifinite projections between smooth algebraic sets is empty or of pure co-dimension 1 .
- In Castle's work over the complex numbers, ACF_0-relics detect multiple intersections, using purity of ramification, one can show that non-locally modular ACF_0-relics are 1-dimensional.

We axiomatize a weaker notion of purity of ramification to get:

Theorem (Castle-Hasson-Y. 2024)

- *K is HGS with enough open maps and purity of ramification, then any strongly minimal non-locally modular K-relic is* 1*-dimensional in the sense of K. Moreover, such relics interprets a strongly minimal group G.*
- ACVF *has purity of ramification.*

Restricted Trichotomy in ACVF

Theorem (Hasson-Onshuus-Pinzon 2024)

Let K be an algebraically closed valued field, and G a non-locally modular strongly minimal K-relic that is a group. If G is locally equivalent to $(K,+)$ or $(K,\cdot)$. Then G interprets a field F which is K-definably isomorphic to $(K,+,\cdot)$. Moreover, the G-induced structure on F is pure ACF.

- With additional work, we could show that the group interpreted in our theorem is locally $(K,+)$ or $(K,\cdot)$ if the the relic is K-definable (instead of K-interpretable).
- Alternatively, one could follow the existing strategy to show trichotomy directly. Namely, purity of ramifications gives that such relics must be dimension 1 and using the tangency to prove trichotomy.

This completes the proof of the restricted trichotomy in the field sorts for ACVF, implying Zilber's Restricted Trichotomy for ACF.

Trichotomy in Imaginaries in ACVF

- What about the conjecture of Kowalski and Randriambololona?
- In recent work of Halevi-Hasson-Peterzil, they classified fields definable in various valued fields. A key machinery is the reduction to four distinguished sorts: Valued field, value group, residue field, and $K/\mathcal{O}$.
- We could follow the same strategy and reduce the question to the distinguished sorts.

Definition

Let D be Γ or $K/\mathcal{O}$. We say D is *locally linear* if whenever $f : U \to D$ is an A-definable function on an open $U \subseteq D^n$, and $a \in U$ with $\mathrm{dprk}(a/A) = n$, there is a neighborhood of a on which f is of the form $x \mapsto g(x) + b$, where $g : D^n \to D$ is a definable group homomorphism and $b \in D$ is a constant.

Our theorem reduces the Kowalski-Randriambololona conjecture to a technical condition on $K/\mathcal{O}$, which we verify in residue characteristic 0.

Theorem (Castle-Hasson-Y. 2024)

Let K be an algebraically closed valued field, and M be a non-locally modular strongly minimal K-relic. If $K/\mathcal{O}$ is locally linear (e.g. when the residue field k has characteristic zero), then M interprets a field F which is K-definably isomorphic to either K or k. Moreover, the M-induced structure on F is pure ACF.

Thank you for your attention!