Upside down and backwards

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Joint work with Tomasz Rzepecki

Introduction

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5. We primarily focus on definably amenable NIP groups.

Outline

Our talk is outlined as follows:

1. Semigroup theory
2. Model theoretic dynamics
3. New results

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The minimal left ideals of left-continuous compact Hausdorff semigroups admit a strong decomposition theorem.

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2. *For each idempotent $u \in I$, $u * I$ is a group.*
3. *For any minimal left ideal J and idempotent $v \in J$ we have that $v * J$ is isomorphic to $u * I$ (as an abstract group). We call $v * J$ an* **Ellis subgroup** *of Y.*

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2. *For each idempotent $u \in I$, $u * I$ is a group.*
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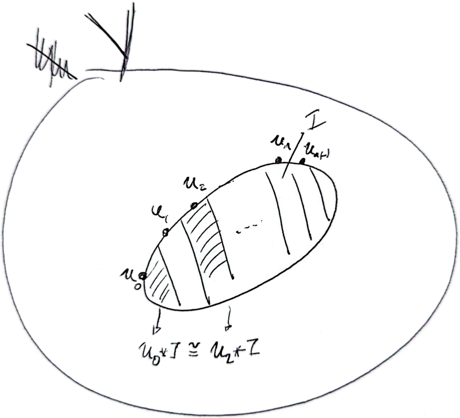
4. *I is the disjoint union of its Ellis subgroups. More explicitly, if $\mathrm{id}(I)$ is the collection of idempotents in I, then*

$$I = \bigsqcup_{u \in \mathrm{id}(I)} u * I.$$

Photograph



Take-away

Take away: Given a left continuous compact Hausdorff semigroup, one can associate a canonical group object, i.e., the isomorphism type of the Ellis subgroups.

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Exercise: $S_x^{\mathrm{fs}}(\mathcal{G}, G) \subseteq S_x^{\mathrm{inv}}(\mathcal{G}, G)$.

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Intuitively, realize q, realize p over the realization of q, then consider the type of the product.

Folklore

Both $S_x^{\text{inv}}(\mathcal{G}, G)$ and $S_x^{\text{fs}}(\mathcal{G}, G)$ with the Newelski product are left-continuous compact Hausdorff groups.

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1. Historical; Newelski originally considered $S_x(G)$, when G is stable; extended to all types definable; then $S_{\text{ext}}(G)$...
2. $S_x^{\text{fs}}(\mathcal{G}, G)$ is isomorphic to an object from classical topological dynamics. Namely, the Ellis semigroup of a particular group action.

Example: Integers

Consider $G = (\mathbb{Z}; +, 0, <)$. Then $S_x^{\mathrm{inv}}(\mathcal{G}, G)$ looks like the following:

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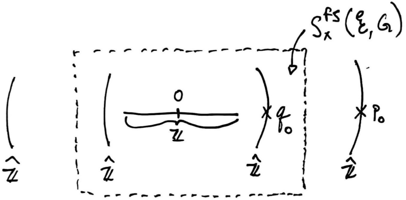
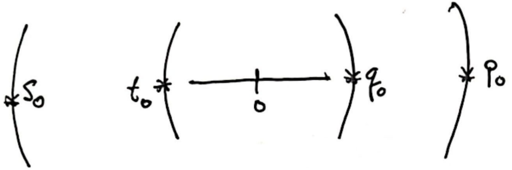


$$p_0 := \{x > a \mid a \in \mathcal{E}\} \cup \{\exists y(\underbrace{y + \_\_ + y}_{n\text{-times}} = x) \mid n \in \mathbb{N}\}.$$

$$q_0 := \{x > a \mid a \in \mathbb{Z}\} \cup \{x < b \mid b > \mathbb{Z}\} \cup \{\exists y(\underbrace{y + \_\_ + y}_{n\text{-times}} = x) \mid n \in \mathbb{N}\}.$$

Example: Integers cont.



$t_0 * q_0 = q_0$ $\qquad$ s_0

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Ideal groups in $S_x^{\mathrm{fs}}(\mathcal{G}, G)$

Recall we are interested in the ideal groups of $S_x^{\mathrm{fs}}(\mathcal{G}, G)$ and $S_x^{\mathrm{inv}}(\mathcal{G}, G)$. The case of $S_x^{\mathrm{fs}}(\mathcal{G}, G)$ has the following history.

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More explicitly, the quotient map $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{G}^{00}$ extends to a map $\hat{\pi} : S_x(\mathcal{G})$ and $\pi|_{u*I} : u * I \to \mathcal{G}/\mathcal{G}^{00}$ is an algebraic isomorphism [where I is a minimal left ideal and u is an idempotent in I].

Definably amenable groups

Fact

Suppose that T is NIP. Then the following are equivalent:

1. *G is definably amenable, i.e., G admits a left invariant measures on the collection of definable subsets.*
2. *$S_x^{\mathrm{inv}}(\mathcal{G}, G)$ admits a left (right) strong f-generic, i.e., there exists some $p \in S_x^{\mathrm{inv}}(\mathcal{G}, G)$ such that every global left (right) translate of p is still an element of $S_x^{\mathrm{inv}}(\mathcal{G}, G)$.*

If G is NIP and definably amenable, we let $\mathcal{F}_r$ be the collection of global right strong f-generics.

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Theorem (G.-Rzepecki 2025+)

Suppose that G is NIP and definably amenable. Then

1. $S_x^{\mathrm{inv}}(\mathcal{G}, G)$ contains a unique minimal left ideal, $\mathcal{F}_r$. These are precisely the *right strong f-generics*.
2. If $u \in \mathcal{F}_r$ is an idempotent, then $u * S_x^{\mathrm{inv}}(\mathcal{G}, G)$ is an ideal group.
3. (HPP) The right stabilizer of any strong right f-generic is $\mathcal{G}/\mathcal{G}^{00}$.
4. For any idempotent $u \in \mathcal{F}_r$, $\pi|_{u * S_x^{\mathrm{inv}}(\mathcal{G}, G)} : u * S_x^{\mathrm{inv}}(\mathcal{G}, G) \to \mathcal{G}/\mathcal{G}^{00}$ is an algebraic isomorphism.

As consequence, we have that the ideal group of $S_x^{\mathrm{inv}}(\mathcal{G}, G)$ and $S_x^{\mathrm{fs}}(\mathcal{G}, G)$ are always isomorphic.

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Ok – So, is there some kind of natural isomorphism? (Reconsider $\mathbb{Z}$)

Retraction

If the underlying theory is NIP, there exists a *mysterious* map

$$F : S^{\mathrm{inv}}_x(\mathcal{G}, G) \to S^{\mathrm{fs}}_x(\mathcal{G}, G)$$

whose definition is a little bit delicate.

Fact (Simon)

The retraction map F from $S^{\mathrm{inv}}_x(\mathcal{G}, G)$ to $S^{\mathrm{fs}}_x(\mathcal{G}, G)$ has the following properties: Let $p, q \in S^{\mathrm{inv}}_x(\mathcal{G}, G)$, then

1. $F(p)|_G = p|_G$,
2. *F is continuous,*
3. *If p is finitely satisfiable in M, then $F(p) = p$,*
4. *For any M-definable function f, $f_*(F(p)) = F(f_*(p))$.*
5. *If q is finitely satisfiable in M, then $F(q_x \otimes p_y) = q_x \otimes F(p_y)$.*

Question: Does the retraction map induce an isomorphism between Ellis subgroups?

It's complicated

Sometimes? It's a little complicated...

Abelian groups

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Lemma

Suppose that G is NIP and definably amenable. Let I be a minimal left ideal contained in $F(\mathcal{F}_r)$. Fix an idempotent $u \in I$. Consider $t \in \mathcal{F}_r$ such that $F(t) = u$. Suppose that every coset of $\mathcal{G}^{00}$ has a representative in G. Then the following are equivalent:

1. *For every $g \in G(M)$, $u \cdot g \in u * I$.*
2. *$F_M|_{t * S_G^{\mathrm{inv}}(\mathcal{U}, M)} : t * S_G^{\mathrm{inv}}(\mathcal{U}, M) \to u * I$ is an isomorphism of Ellis subgroups.*

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Lemma

Suppose that G is NIP and definably amenable. Let I be a minimal left ideal contained in $F(\mathcal{F}_r)$. Fix an idempotent $u \in I$. Consider $t \in \mathcal{F}_r$ such that $F(t) = u$. Suppose that every coset of $\mathcal{G}^{00}$ has a representative in G. Then the following are equivalent:

1. *For every $g \in G(M)$, $u \cdot g \in u * I$.*
2. *$F_M|_{t * S_G^{\mathrm{inv}}(\mathcal{U},M)} : t * S_G^{\mathrm{inv}}(\mathcal{U}, M) \to u * I$ is an isomorphism of Ellis subgroups.*

Theorem

Suppose G is NIP, abelian, and G contains representatives for each coset of $\mathcal{G}^{00}$. Let I be a minimal left ideal of $F(\mathcal{F}_r)$. Fix an idempotent $u \in I$. Consider $t \in \mathcal{F}_r$ such that $F_M(t) = u$. Then the map $F_M|_{t * S_G^{\mathrm{inv}}(\mathcal{U},M)} : t * S_G^{\mathrm{inv}}(\mathcal{U}, M) \to u * I$ is an isomorphism of Ellis subgroups.

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The retraction map is not an isomorphism of Ellis groups in $\mathbb{R} \rtimes \{\pm 1\}$.

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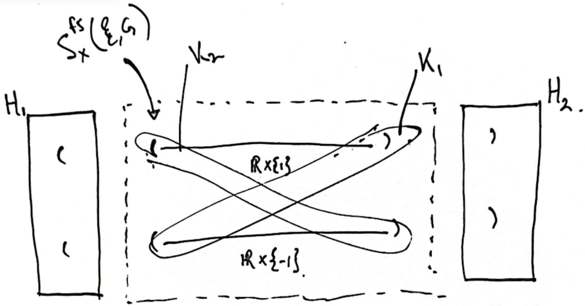
$$(a, i) \cdot (b, j) = \begin{cases} (a + b, j) & \text{if } i = 1, \\ (a - b, -j) & \text{if } i = -1. \end{cases}$$

$\mathbb{R} \rtimes \{\pm 1\}$



* H_1, H_2 are ideal subgroups of $S_x^{inv}(\xi, G)$.

* K_1, K_2 are ideal subgroups of $S_x^{fs}(\xi, G)$

Upside down and backwards

Consider the model theoretic inversion, ${}^{-1} : S_x(\mathcal{G}) \to S_x(\mathcal{G})$ defined via $p^{-1} = \mathrm{tp}(a^{-1}/\mathcal{G})$.

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Definition

We say that a definably amenable G is dfg if there exists some $p \in S_x^{\mathrm{inv}}(\mathcal{G}, G)$ such that p is definable over G and every global translate of p is also definable over G.

Intuition: dfg groups are the opposite of compact.

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Theorem (G.-Rzepecki 2025+)

Suppose that T is NIP, G is dfg, and t is a right dfg type over M. Then $F \circ^{-1} |_{t * S_G^{\mathrm{inv}}(\mathcal{U}, M)}$ is an anti-isomorphism from an invariant Ellis subgroup to a finitely satisfiable Ellis subgroup. Precomposing with group inversion give an honest-to-goodness isomrophism.

Thank you

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