A variant of continuous Chaitin's Ω function

Yuxuan Li   Shuheng Zhang   Xiaoyan Zhang   Xuanheng Zhao

September 5, 2025

Table of Contents

Table of Contents

Definition

A set $A \subseteq 2^{<\omega}$ is prefix-free if it is an antichain with respect to the natural partial order on $2^{<\omega}$.

Randomness

Definition

A set $A \subseteq 2^{<\omega}$ is prefix-free if it is an antichain with respect to the natural partial order on $2^{<\omega}$.

Definition

A prefix-free machine is a Turing machine whose domain is prefix-free.

Randomness

Definition

A set $A \subseteq 2^{<\omega}$ is prefix-free if it is an antichain with respect to the natural partial order on $2^{<\omega}$.

Definition

A prefix-free machine is a Turing machine whose domain is prefix-free.

Definition

The Kolmogorov complexity of a string σ with respect to a Turing machine M is

$$K_M(\sigma) = min(\{|\tau| : M(\tau) = \sigma\}).$$

Randomness

Definition

A prefix-free machine U is optimal if for any prefix-free machine M there is a constant c such that for any string σ

$$K_U(\sigma) \leq K_M(\sigma) + c.$$

We fix a optimal prefix-free machine U and if there is no ambiguity, the Kolmogorov complexity $K(\sigma)$ of a string σ denotes $K_U(\sigma)$.

Randomness

Definition

A prefix-free machine U is optimal if for any prefix-free machine M there is a constant c such that for any string σ

$$K_U(\sigma) \leq K_M(\sigma) + c.$$

We fix a optimal prefix-free machine U and if there is no ambiguity, the Kolmogorov complexity $K(\sigma)$ of a string σ denotes $K_U(\sigma)$.

Definition(Chaitin)[**chaitin˙1975**]

A real $x \in 2^\omega$ is $1-$*random* if there is a constant c such that

$$\forall n K(A \upharpoonright n) \geq n + c.$$

Chaitin's Ω

Definition(Chaitin)

For an optimal prefix-free machine U, we define the Chaitin's Ω relative to U as

$$\Omega_U = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|}.$$

Chaitin's Ω as a function

Downey[**downey2005relativizing**]

Define Ω_U from 2^ω to 2^ω as

$$\Omega_U(x) = \sum_{U^x(\sigma)\downarrow} 2^{-|\sigma|}.$$

Chaitin's Ω as a function

Downey[**downey2005relativizing**]

Define Ω_U from 2^ω to 2^ω as

$$\Omega_U(x) = \sum_{U^x(\sigma)\downarrow} 2^{-|\sigma|}.$$

Becher and Grigorieff[**Becher˙05**]

Define Ω_U from $P(\mathbb{N})$ to 2^ω as

$$\Omega_U(O) = \sum_{\sigma \in U^{-1}(O)\downarrow} 2^{-|\sigma|}.$$

Chaitin's Ω as a function

Hölzl, Merkle, Miller, Stephan and Yu [hlzl˙merkle˙miller˙stephan˙yu˙2020]

Define $\hat{\Omega}_U$ from 2^ω to 2^ω as

$$\hat{\Omega}_U(x) = \sum_{\sigma \prec x} 2^{-K_U(\sigma)}.$$

Chaitin's Ω as a function

Hölzl, Merkle, Miller, Stephan and Yu [**hlzl˙merkle˙miller˙stephan˙yu˙2020**]

Define $\hat{\Omega}_U$ from 2^ω to 2^ω as

$$\hat{\Omega}_U(x) = \sum_{\sigma \prec x} 2^{-K_U(\sigma)}.$$

Zhang[**Zhang**]

Define f_U from 2^ω to 2^ω as

$$f_U(x) = \sum_{\sigma \leq_L x} 2^{-K_U(\sigma)}.$$

Table of Contents

The differentiability of f

Definition

A real $x \in 2^\omega$ is density random if x is 1-random and has density 1 in every Π^0_1 class containing x.

Theorem (Miyabe, Nies and Zhang [**miyabe˙nies˙zhang˙2016**])

x is density random if and only if $g'(x)$ exists for all interval-c.e. function g.

Lemma[**hlzl˙merkle˙miller˙stephan˙yu˙2020**]

If x is 1-random, then

$$\lim_{n\to\infty} 2^n \sum_{m\geq 0} 2^{-K((x\restriction n)0^m)} = 0.$$

The differentiability of F

Theorem

Define $F : [0,1] \to [0,1]$ as $F(x) = f(B(x))$ where $B(x)$ is the infinite binary expression with infinitely many 0. A real x is density random if and only if f is differentiable at x. In this case $F'(x) = 0$.

Proof Sketch

$\Rightarrow$: By theorem above.

$\Leftarrow$: Similar to [**hlzl˙merkle˙miller˙stephan˙yu˙2020**].

- If x is not 1-random, then F is not differentiable at x.
- Suppose that F is differentiable at x, then $F'(x) = 0$.
- If x is not density random, then F is not differentiable at x.

The image of f

Proposition

$f(2^\omega)$ is null, nowhere dense and perfect Π^0_1 relative to $\emptyset'$ class.

Figure: The image of f

corollary

For any x, $f(x)$ is not *weakly* 1-*random* relative to $\emptyset'$.

Hausdorff Dimension

Definition (Hausdorff Measure)

For $A \subseteq 2^\omega$, the s-dimensional outer Hausdorff measure is:

$$\mathcal{H}^s_n(A) = \inf\left\{\sum_{\sigma\in D} \mu_s([\sigma]) : D \subseteq 2^{\geq n},\ A \subseteq [D]\right\}$$

$$\mathcal{H}^s(A) = \lim_{n\to\infty} \mathcal{H}^s_n(A)$$

where $\mu_s([\sigma]) = 2^{-s|\sigma|}$.
The Hausdorff dimension of A is:

$$\dim_H(A) = \inf\{s : \mathcal{H}^s(A) = 0\}$$

Generalized Cantor Sets

Definition

A generalized cantor sets with scale γ is 2^ω with middle $\frac{1}{\gamma}$ of each interval removed iteratively:

$$C_0^\gamma = [0,1]$$
$$C_n^\gamma = \frac{\gamma-1}{2\gamma} C_{n-1}^\gamma \cup \left(\frac{\gamma+1}{2\gamma} + \frac{\gamma-1}{2\gamma} C_{n-1}^\gamma \right)$$
$$C^\gamma = \bigcap_n C_n^\gamma$$

Fact

$$\dim_H(C^\gamma) = -\frac{\log 2}{\log\left(\frac{\gamma-1}{2\gamma}\right)},$$

and as $\gamma \to \infty$, $\dim_H(C^\gamma) \to 1$.

$\dim_H(f[2^\omega]) = 1$

Theorem

The image set $f[2^\omega] = \{f(x) : x \in 2^\omega\}$ has Hausdorff dimension 1.

Proof Sketch

1. Construct maps satisfying Lipschitz condition $g_n : [0,1] \to [0,1]$ with uniform constant c,
2. Define limit function $g(x) = \lim_{n\to\infty} g_n(x)$ which also satisfies Lipschitz condition,
3. Apply Mapping Theorem:

$$|g(x) - g(y)| \leq c|x - y| \implies \mathcal{H}^s(g(A)) \leq c^s \mathcal{H}^s(A)$$

Table of Contents

Algorithmic properties

Proposition: Turing Computability Relations

Given a real x. (i) $x' \geq_T \emptyset' \oplus x \geq_T f(x)$; (ii) $f(x)' \geq_T \emptyset' \oplus f(x) \geq_T x$; (iii) $f(x) \oplus x \geq_T \emptyset'$.

Proof.

(i) Given $f(x)$, $\emptyset'$ can decide wether $\sigma \leq_L x$.
(ii) Similar to (i).
(iii) For almost all n, if $f(x) - f_s(x) < 2^{-8^n}$ at stage $s > 4^n$, then $n \in \emptyset'$ if and only if $n \in \emptyset'_{s+1}$. □

Algorithmic properties

Definition(Miller)[**miller2006contrasting**]

A real x is *weakly low for K* if

$$\exists^\infty n(K(n) \leq K^x(n) + O(1))$$

Definition(Hölzl, et al)[**hlzl˙kraling˙merkle˙2009**]

A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is a Solovay function relative to A, if f is right c.e. relative to A, $K^A(n) \leq f(n) + c$ for some constant c, and for some d, $f(n) \leq K^A(n) + d$ for infinitely many n.

Theorem(Hölzl, et al)[**hlzl˙kraling˙merkle˙2009**]

A right-c.e. function f is a Solovay function relative to X if and only if $\sum_n 2^{-f(n)}$ is ML-random relative to X.

Algorithmic properties

Theorem

A real $x \neq 0$ is *weakly low for K*, if and only if $f(x)$ is $x-random$.

proof sketch

Let s be the least number such that $0^s1 \leq_L x$. Define g from ω to ω as:

$$g(n) = \begin{cases} n, & n <_L 0^s1 \\ e(n), & n \geq_L 0^s1 \end{cases}$$

where $e(n)$ is a computable permutation from $\{n : n \geq_L 0^s1\}$ to $\{n : n \geq_L 0^s1 \wedge n <_L x\}$.

Algorithmic properties

Corollary

For all *weakly low for K* but not *K-trivial* x:

$$f(x) \not\geq_T \emptyset'$$

f is not $Turing-invariant$

Theorem(with Slaman)

There are x, y such that $x \equiv_T y$ and $f(x) \not\equiv_T f(y)$.

Proof Sketch

Suppose for all $x \equiv_T y$ we have $f(x) \equiv_T f(y)$. Note that for all x, x is $right-c.e.$ to $f(x)$, So $\bar{x}$ is $right-c.e.$ to $f(\bar{x})$. Hence for all x, $f(x) \geq_T x$ which is a contradiction.

Image of f

Theorem 2(with Yu)

There are uncountbly many x such that $f(x)$ is not random. Moreover $\{x : f(x) \text{ is not } 1 - \textit{random}\}$ is null.

Image of f

Theorem 2(with Yu)

There are uncountbly many x such that $f(x)$ is not random. Moreover $\{x : f(x) \text{ is not } 1 - random\}$ is null.

Small perturbation lemma

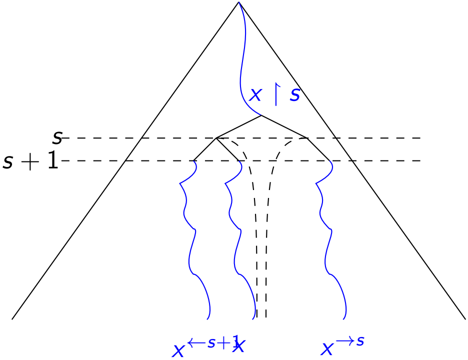
For all real x and $n \in \omega$, if there exists j such that $|f(x^{\triangle j}) - f(x)| > 2^{-n}$, then there is $y \in 2^{\omega}$ such that $2^{-n-c} \leqslant |f(y) - f(x)| \leqslant 2^{-n}$.

Small perturbation lemma

Small perturbation lemma

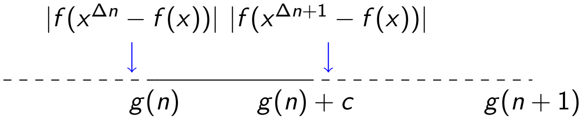
For all real x and $n \in \omega$, if there exists j such that $|f(x^{\triangle j}) - f(x)| > 2^{-n}$, then there is $y \in 2^\omega$ such that $2^{-n-c} \leqslant |f(y) - f(x)| \leqslant 2^{-n}$.

Figure: the intuition of some case of small perturbation lemma



Proof of Theorem 2

Figure: the intuition of theorem 2



Proof.

Define $g(0) = 0$ and $g(n+1) = g(n) + n + c$.
We use small perturbation lemma to make sure that if $f(x) \upharpoonright [g(n-1)+c, g(n)] \neq 1^{n+1}$ and $f(x) \upharpoonright [g(n-1)+c, g(n)) \neq 0^n$ then $f(x) \upharpoonright [g(n), g(n)+c) \neq 0^c$.
Since all $2-\mathit{random}$ real is $\mathit{weakly\ low\ for\ K}$ and

$$\{x : f(x) \text{ is not } 1-\mathit{random}\} \subsetneq \{x : x \text{ is not } 2-\mathit{random}\}.$$

$\square$

Table of Contents

Questions

Question 1

If x is not K-trivial, can f be Turing invariant on $deg(x)$?

Question 2

Is there a computable real in $f(2^\omega)$? Given a computable real (or just a rational) p, is there is an optimal prefix-free machine V such that $p \in f_V(2^\omega)$?

Questions

Proposition(with Slaman)

If $f(x)$ is right-c.e., then x must be non-1-random and right-c.e.. Moreover, if $f(x)$ is computable, then x is also Turing complete.

Proof.

Given right-c.e. q with approximation $(q_s)_{s\in\omega}$, we construct some opponent machine M_q with coding constant c_M, that is, $\forall\sigma K(\sigma) \leq K_M(\sigma) + c_M$, and use M to attack the optimality of U to make sure $\forall x f(x) \neq q$. By recursion theorem, c_M can be used in the construction of M. First, define $J_\sigma = (f(\sigma 0^\infty), f(\sigma 1^\infty))$ and $J_{\sigma,s} = (f_s(\sigma 0^\infty), f_s(\sigma 1^\infty))$. Whenever q_s in some small interval $J_{\sigma,s}$, M gives a short description of some string τ on the left of σ to force U give a short description of τ later, which implies $f(\sigma 0^\infty) - f_s(\sigma 0^\infty)$ is big enough to make sure $q \notin J_\sigma$.

$\square$

[1] George Barmpalias.
Aspects of Chaitin's Omega. in
Algorithmic randomness—progress and prospects.
Cambridge Univ. Press, Cambridge, 175–205, 2020.

[2] Verónica Becher, Santiago Figueira, Serge Grigorieff and Joseph S. Miller.
Randomness and halting probabilities.
J. Symbolic Logic, 71(4):1411–1430, 2006.

[3] L. Bienvenu and R. G. Downey, Kolmogorov complexity and Solovay functions, in *STACS 2009: 26th International Symposium on Theoretical Aspects of Computer Science*, 147–158, LIPIcs. Leibniz Int. Proc. Inform., 3, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2009.

[4] Laurent Bienvenu, Noam Greenberg, Antonín Kučera, André Nies and Dan Turetsky.
Coherent randomness tests and computing the K-trivial sets.
J. Eur. Math. Soc. (JEMS), 18(4):773–812, 2016.

[5] Gregory J. Chaitin.
A theory of program size formally identical to information theory.
J. ACM., 22(3):329–340, 1975.

[6] Adam R. Day and Joseph S. Miller.
Density, forcing, and the covering problem.
Math. Res. Lett., 22(3):719–727, 2015.

[7] Rodney G. Downey and Denis R. Hirschfeldt.
Algorithmic randomness and complexity.
2010.

[8] Rod Downey, Denis R. Hirschfeldt, Joseph S. Miller, and André Nies.
Relativizing Chaitin's halting probability.
J. Math. Log., 5(2):167–192, 2005.

[9] Rupert Hölzl, Thorsten Kräling, and Wolfgang Merkle.
Time-Bounded Kolmogorov Complexity and Solovay Functions.
Theory of Computing Systems, 52:392–402, 2009.

[10] Rupert Hölzl, Wolfgang Merkle, Joseph Miller, Frank Stephan, and Liang Yu.
Chaitin's Ω as a continuous function.
J. Symb. Log., 85(1):486–510, 2020.

[11] M. Khan. Lebesgue density and Π^0_1 classes. *J. Symb. Log.*, 81(1):80–95, 2016.

[12] A. Kučera, and T. A. Slaman.

Randomness and recursive enumerability.
SIAM Journal on Computing, 31:199–211, 2001.

[13] P. Mattila.
Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability.
Cambridge University Press, 1995.

[14] Joseph S. Miller.
The K-Degrees, Low for K Degrees, and Weakly Low for K Sets.
Notre Dame J. Formal Log., 50:381–391, 2009.

[15] Kenshi Miyabe, André Nies, and Jing Zhang.
Using almost-everywhere theorems from analysis to study randomness.
Bull. Symb. Log., 22(3):305–331, 2016.

[16] Keng Meng Ng, Frank Stephan, Yue Yang, and Liang Yu.
Computational aspects of the hyperimmune-free degrees. in
Proceedings of the 12th Asian Logic Conference.
World Sci. Publ., Hackensack, NJ, 271–284, 2013.

[17] André Nies.
Computability and randomness, volume 51 of *Oxford Logic Guides*.
Oxford University Press, Oxford, 2009.

[18] Robert Rettinger and Xizhong Zheng.
Solovay reducibility on d.c.e. real numbers.
In *International Computing and Combinatorics Conference*, 359–368, 2005.

[19] Robert I. Soare.
Recursively enumerable sets and degrees: A study of computable functions and computably generated sets, *Perspectives in Mathematical Logic*.
Springer-Verlag, Berlin, 1987.

Thanks!