Forcing construction related to club principle

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$\diamondsuit$ (a.k.a. Diamond principle) is one the the most well studied set-theoretical combinatorial properties.

For uncountable regular cardinal κ, we say a sequence $\langle a_\alpha \mid \alpha \in \kappa \rangle$ is a $\diamondsuit_\kappa$ sequence if

- For any α, $a_\alpha \subseteq \alpha$.
- For each $A \subseteq \kappa$, there are stationary many α such that $A \cap \alpha = a_\alpha$.

Then we say $\diamondsuit_\kappa$ holds if there is a $\diamondsuit_\kappa$ sequence.

$\diamondsuit$ is introduced by Jensen.

- If $V = L$, then for any regular κ, $\diamondsuit_\kappa$ holds.
- $\diamondsuit_{\omega_1}$ implies the existence of Suslin tree on ω_1 and hence the failure of Suslin Hypothesis.

Meanwhile $\diamondsuit$ is a fundamental tool in the consistency argument.

Example of construction using $\diamondsuit$:

- Algebra: non-free Whitehead groups(Shelah)
- Topological spaces: ω_1-size Dowker space.
- Application in order, graph, Banach space, operational algebra...

A natural weakening of $\diamondsuit$ is Continuum Hypothesis.

- $\diamondsuit_{\kappa^+}$ implies $2^\kappa = \kappa^+$.

  For any $A \subseteq \kappa$, there is a a_{α_A} such that $a_{\alpha_A} \cap \kappa = A$, hence $A \mapsto \alpha_A$ defines a injection from $\mathcal{P}(\kappa)$ to κ^+.
- For uncountable κ, $2^\kappa = \kappa^+$ implies $\diamondsuit_{\kappa^+}$. (Shelah)
- On ω_1, CH is equivalent to weak diamond principle, a weak form of $\diamondsuit$. (Delvin-Shelah)
- A model with CH but no Suslin tree. (Jensen)

Club Principle ($\clubsuit$)

There are many other weakenings of $\diamondsuit$. We focus on $\clubsuit$ (a.k.a. Club Principle) introduced by Ostaszewski. It is the counterpart of CH under $\diamondsuit$.

For uncountable regular cardinal κ, we say a sequence $\langle a_\alpha \mid \alpha \in \kappa \rangle$ is a $\clubsuit_\kappa$ sequence if:

- For any α, a_α is an unbounded subset of α.
- For unbounded $A \subseteq \kappa$, there are stationary many α such that $a_\alpha \subseteq A$.

Then we say $\clubsuit_\kappa$ holds if there is a $\clubsuit_\kappa$ sequence.

- $CH + \clubsuit_{\omega_1} \iff \diamondsuit_{\omega_1}$. (Devlin, Burgess)

Further Results on $\clubsuit_{\omega_1}$

- Shelah constructed a model with $\neg CH + \clubsuit_{\omega_1}$, thus separating $\diamondsuit_{\omega_1}$ and $\clubsuit_{\omega_1}$.
- Baumgartner gave another construction, without collapsing ω_1.
- Con(MA(countable) $+ \clubsuit_{\omega_1}$). (Kojman, Fuchino-Shelah-Soukup)
- Further techniques developed by Shelah-Dzamonja, Shelah-Mildenberger.
- Woodin's $\mathbb{Q}_{\max}^{\clubsuit_{NS}}$ model satisfies a variant of $\clubsuit_{\omega_1}$.

$\clubsuit_{\omega_1}$ and its consequences

$\clubsuit_{\omega_1}$ implies some consequences of CH:

- $\clubsuit_{\omega_1}$ gives an ω_1-size Dowker space.
- $\clubsuit_{\omega_1}$ implies $\mathrm{add}(\mathcal{N}) = \omega_1$. (Truss)
- $\clubsuit_{\omega_1} + \mathrm{cof}(\mathcal{M}) = \omega_1$ implies there is a Suslin tree. (Brendle)
  - A tree on $\omega^{<\omega_1}$ is Suslin if any antichain of the tree is countable.

The purpose to develop various forcing constructions related to $\clubsuit_{\omega_1}$ is to give models with consequence of $\diamondsuit_{\omega_1}$ and $\neg$CH.

Open Questions on $\clubsuit_{\omega_1}$

Juhasz poses the following question:

- Does $\clubsuit_{\omega_1}$ imply the existence of a Suslin tree?

It is a natural question, though it appear to be quite difficult. Shelah gave two false proof attempts. Juhasz's question is a major open problem in the forcing theory.

Similar open questions include:

- $\mathrm{Con}(\clubsuit_{\omega_1} + \mathfrak{s} > \omega_1)$?
- $\mathrm{Con}(\clubsuit_{\omega_1} + \mathfrak{h} > \omega_1)$?
- $\mathrm{Con}(\clubsuit_{\omega_1} + \mathrm{add}(\mathcal{M}) > \omega_2)$?

New Forcing Construction

We intend to give a new forcing construction which might be useful to tackle Juhasz's question, which is completely open now.

Known constructions:

- Shelah's pseudo product (itearted) forcing.
- Countable support iteration with Mildenberger's finite condition proper forcing.

pseudo product

Let $\prod_{i\in\lambda}^{*} \mathrm{Add}(\omega_1)$ be the set:

$$\left\{ p \in \prod_{i\in X} \mathrm{Add}(\omega_1) : |\mathrm{supp}(p)| < \omega_1 \right\}$$

with the partial ordering:

$p \leq q \iff p(i) \leq q(i)$ for all $i \in X$ and $\{i \in X : p(i) \not\leq q(i) \neq 1_{P_i}\}$ is finite.

$\prod_{i\in\lambda}^{*} \mathrm{Add}(\omega_1)$ forces $\clubsuit_{\omega_1}$, and $2^{\omega} = \lambda$.

Forcing with Model as Side Condition

In 1980s, Todorcevic invented the forcing with model as side condition technique. He introduced a toy example which was somehow ignored later, until 30 years later Aspero-Mota rediscovered it.

The forcing is defined by:

- κ is uncountable regular.
- Condition p is a finite symmetry set of countable models $\mathcal{M} \prec H(\kappa)$.
- Order by inclusion.

Here a set of models p is symmetric the following is true:

- For all $N, N' \in p$ and all $\xi \in N \cap \beta$, if $\delta_N = \delta_{N'}$, then there is a unique isomorphism $\pi_{N,N'}$ between N and N'.
- For all $N_0, N_1 \in p$, if $\delta_{N_0} < \delta_{N_1}$, then there is some $N_2 \in p$ such that $\delta_{N_2} = \delta_{N_1}$ and $N_0 \in N_2$.
- For all $N_0, N_1, N_2 \in \mathcal{N}$, if $N_0 \in N_1$ and $\delta_{N_1} = \delta_{N_2}$, then $\pi_{N_1,N_2}(N_0) \in p$.

This forcing forces CH. The amazing thing is that this forcing actually forces $\diamondsuit_{\omega_1}$. (See some details on the board.)

Aspero-Mota then defines an iterated forcing using the Todorcevic forcing as a skeleton. We also give a toy example as follows:

- κ is uncountable regular.
- Condition p is a finite symmetry set of countable models $\mathcal{M} \prec H(\kappa)$ and a finite partial function $f : \kappa \times \omega \to 2$ satisfying the symmetry condition:
  - For any $\mathcal{M} \cong \mathcal{N}$ in p as models, $\langle \mathcal{M}, p \cap \mathcal{M} \rangle \cong \langle \mathcal{N}, p \cap \mathcal{N} \rangle$.
- Order by inclusion.

Again, this forcing forces $\diamondsuit_{\omega_1}$. However, notice that each $f_\alpha : \omega \to 2$ is a new real. We add many new Cohen reals to ground model.

Main Idea: Making f_α Distinct

Main idea: How about deliberately making all f_α distinct, but requiring them to still guess enough information about subsets of ω_1?

From now on, we work in L. We assume that there is a seuqence $\langle M_\alpha, A_\alpha \rangle$ which guess all ω_1-size model $\langle \mathcal{M}, \mathcal{A} \rangle$ at stationary many α.

Recall that a set of conditions A is a Δ-system of $\mathsf{Add}(\omega, \omega_2)^M$, if $A \subseteq \mathsf{Add}(\omega, \omega_2)$ and there is an $r \in \mathsf{Add}(\omega, \omega_2)^M$ such that for all $p, q \in A$, $\mathrm{dom}(p) \cap \mathrm{dom}(q) = \mathrm{dom}(r)$ and $p \restriction \mathrm{dom}(r) = r$.

For any α, we say $(-)_\alpha$ holds if X_α codes a ZF- model $\langle M_\alpha, A_\alpha \rangle$ where A_α is a Delta-system in $Add(\omega, \omega_2)^{M_\alpha}$. Let r_α be the root of A_α. Let $\tilde{A}_\alpha = \langle \tilde{a}_\gamma \mid \gamma < \omega_1^{M_\alpha} \rangle$, where $\tilde{a}_\gamma = a_\gamma \restriction (\mathrm{dom}(a_\gamma) \setminus \mathrm{dom}(r_\alpha))$. Finally, we fix an auxiliary ω-length subsequence $\bar{A}_\alpha$ of $\tilde{A}_\alpha$ such that the index set of $\bar{A}_\alpha$ is cofinal in $\omega_1^{M_\alpha}$. We remark here we allow the trivial case that all elements of $\tilde{A}_\alpha$ are emptyset.

A Condition in $\mathbb{P}$

A condition in $\mathbb{P}$ is of the form $p = (f_p, \Delta_p)$, where

1. $f_p : \omega_2 \to \omega^{<\omega}$ is a finite partial function.
2. Δ_p is a sequence of symmetric system of countable substructure of L_{ω_2}.
3. (Symmetry Condition) Suppose N_1, N_2 are two isomorphic models in Δ_p. Suppose $(-)_{\delta_{N_i}}$ holds and both of N_i are isomorphic to $H(\omega_2)^{X_{\delta_N}}$. Then for any $a \in \bar{A}_{\delta_{N_i}}$,

$$a^{N_1} \subseteq f_p \cap N_1 \text{ if and only if } a^{N_2} \subseteq f_p \cap N_2.^{1} \qquad (*)$$

[1] Note if one of $\mathrm{dom}(a_i) \cap \mathrm{dom}\, f_p$ is non-empty, and $a_i \not\subseteq f_p$, then $\mathrm{dom}(a_{3-i}) \subseteq \mathrm{dom}(f_p)$ and $a_{3-i} \not\subseteq f_p$.

Defining the Guessing Sequence

The main point is that we can define the guessing sequence.

In the final model, fix any α, we define c_α as follows:

Suppose for some $X[G] \prec H(\kappa)[G]$ and the type of M_{δ_X} is guessed. Suppose $r_M \subseteq f_G$. Suppose M_{δ_X} moreover thinks the following holds:

1. $\langle \tau_\alpha \mid \alpha < \omega_1 \rangle$ is a sequence of names of ordinals in ω_1 such that $\Vdash_{\mathbb{P}} \tau_\alpha \neq \tau_\beta$ for any distinct $\alpha, \beta < \omega_1$.
2. A_{δ_X} is a Δ-system.
3. for any γ, there are dense many conditions p of $\mathbb{P}$ such that there is $\epsilon > \gamma$, $a_\epsilon \in A_{\delta_X}$ and q such that $f_q = a_\epsilon$, q compatible with p and decides τ_ϵ.

Then define c^X as the following set:

$$\{\gamma < \delta_X \mid \exists \epsilon \in \delta_X \exists p \in G \cap X (f_p = a^X_\epsilon \wedge p \models A_\beta = \gamma)\}.$$

Claim

Claim

If $X \cong X'$ are such that c^X and $c^{X'}$ are defined, then $c^X = c^{X'}$.

Hence, we can define c_α to be some c^X when c^X is defined and $\alpha = \delta_X$.

Iterated Forcing and Application

Now we can proceed to iterated forcing. The construction is more complicated. As an application, we give a model of $\clubsuit_{\omega_1} + \mathrm{add}(\mathcal{M}) > \omega_2$.

As in the original construction of Aspero-Mota, we need a stronger version of Symmetrity varing along iteration.

For any α, we say $(-)_\alpha$ holds if X_α codes a ZF-model $\langle M, A, B \rangle$, where A is a Δ-system in $\mathrm{Hechler}(\kappa)^M$. We denote such A by A^M. We write $\tilde{A}^M = \langle \tilde{a}_\alpha \mid \alpha < \omega_1 \rangle$, where:

$$\tilde{a}_\alpha = a_\alpha \restriction (\mathrm{dom}(a_\alpha) \setminus \mathrm{dom}(r^M)),$$

and r^M is the root of A^M.

A Condition in $\mathbb{P}$

Assume that for some $\beta < \kappa$, all previous $\mathbb{P}_\alpha$ is defined:

We define $\mathbb{P}_\alpha$ as following: Assume first that $\beta < \kappa$. Conditions in $\mathbb{P}_\beta$ are pairs of the form $q = (p, \Delta)$ with the following properties:

(C0) p is a finite function such that $\mathrm{dom}(p) \subseteq \beta$ and Δ is a set of pairs (N, γ) with $\gamma \leq \beta$.

(C1) $\Delta^{-1}(\beta)$ is a $\mathbb{T}^{\beta+1}$-symmetric system.

(C2) For every $\alpha < \beta$, the restriction of q to α is a condition in $\mathbb{P}_\alpha$. This restriction is defined as:

$$q_\alpha := (p \restriction \alpha, \{(N, \min\{\alpha, \gamma\}) : (N, \gamma) \in \Delta\}).$$

(C3) If $\alpha \in \mathrm{dom}(p)$, then $p(\alpha)$ is a Hechler condition over $\mathbb{P}_\alpha$.

(C4) (Symmetry Condition) Suppose N_1, N_2 are two isomorphic models in $\Delta^{-1}(\beta)$. Let N be their common image under a transitive collapsing mapping. Suppose $(-)^{\beta}_{\delta^N}$ holds and N is the domain of X_{δ^N}. Let A_1, A_2 be the corresponding images of A^N_δ in N_1 and N_2. Fix any $a \in \bar{A}_\alpha$, and let a_1, a_2 be the images of a. Then:

$$a_1 \subseteq f_p \cap N_1 \text{ if and only if } a_2 \subseteq f_p \cap N_2.$$

Here the definition of $(-)^{\beta}_{\delta^N}$ is similar to the product case but incorporates the knowledge of previous iteration.

Given conditions $q_\epsilon = (p_\epsilon, \Delta_\epsilon)$ (for $\epsilon \in \{0, 1\}$) in $\mathbb{P}_\beta$, we will say that $q_1 \leq_\beta q_0$ if and only if the following holds:

(D1) $q_1 \restriction \alpha \leq_\alpha q_0 \restriction \alpha$ for all $\alpha < \beta$,

(D2) $\mathrm{dom}(p_0) \subseteq \mathrm{dom}(p_1)$,

(D3) $f^{p_0,\alpha} \subseteq f^{p_1,\alpha}$,

(D4) $\Delta_0^{-1}(\beta) \subseteq \Delta_1^{-1}(\beta)$.

Some arguments shows that the forcing preserves cardinality. The Helcher reals added along the iteration increase *add*$(\mathcal{M})$. We can also show that $\clubsuit_{\omega_1}$ holds in the final model.

Back to Juhasz's Question

Back to Juhasz's question. One can try to naturally extend the above construction using finite specializing tree iteration. The main obstacle here is the conflict between the symmetry requirement and the information coming from higher nodes of the tree. We don't know how to overcome it right now.

Thank you for your attention!