# On successive categoricity, stability and NIP in abstract elementary classes


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## Abstract

This dissertation covers several results in classification theory of abstract elementary classes. In the third and the fourth chapters we deal with cases of the following conjecture studied extensively by Shelah under additional model theoretic and set theoretic assumptions: does $\mathbb{I}(\mathbf{K},\lambda) = \mathbb{I}(\mathbf{K},\lambda^+) = 1$ imply $\mathbf{K}_{\lambda^{++}} \neq \emptyset$? 1

**Theorem 1.** *Suppose $\lambda^+ < 2^{\aleph_0}$. Assume $\mathbf{K}$ has amalgamation in $\lambda$, no maximal model in $\lambda$, and is stable for $\lambda$-algebraic types in $\lambda$. If $\mathbf{K}$ is $(<\lambda^+,\lambda)$-local, then $\mathbf{K}$ has a model of cardinality $\lambda^{++}$.*

Also in the fourth chapter we prove that under certain assumptions, existence of a model in $\lambda^{++}$ implies stability in $\lambda$. We also show that stability in $\lambda$ and existence in $\lambda^{++}$ are equivalent.

**Theorem 2.** *Suppose $\lambda^+ < 2^\lambda < 2^{\lambda^+}$. Assume $\mathbf{K}$ is categorical in $\lambda$ and $\lambda^+$, $\mathbf{K}$ is $(<\lambda^+,\lambda)$-local and $\mathbf{K}$ is almost stable in $\lambda$. The following are equivalent.*

1. *$\mathbf{K}$ has a model in $\lambda^{++}$.*
2. *$\mathbf{K}$ is stable in $\lambda$.*

In the fifth chapter we introduce and investigate a parallel of NIP (dependent theories) for abstract elementary classes. In particular we show that the negation of it leads to being able to encode subsets.

**Theorem 3.** *Suppose $\mathbf{K}$ is $(<\aleph_0)$-tame, $M \in \mathbf{K}$, $C \subseteq |M|$, $\lambda := |C| \geq \beth_3(LS(\mathbf{K}))$ and $(ded\,\lambda)^{2^{LS(\mathbf{K})}} = ded\,\lambda$. Suppose $|\mathbf{S}^1(C;M)| > ded\,\lambda$. Then there is $N \in \mathbf{K}$, $\langle \bar{a}_n \in {}^m|N| \mid n < \omega\rangle$ and $\phi$ in the language of the Galois Morleyization of $\mathbf{K}$ such that for every $w \subseteq \omega$ there is $b_w \in |N|$ such that for all $i < \omega$,*

$$N \models \phi(\bar{a}_i, b_w) \iff i \in w.$$

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# Contents

# Chapter 1

# Introduction

This thesis presents some results in the study of abstract elementary classes. The topics include approximations to the successive categoricity question, stability, and analogues of NIP (in elementary classes) in abstract elementary classes.

Model theory is the study of certain classes of mathematical structures, usually those that are axiomatized by a set of logical formulas in a fixed logic. Since the early developments of the field, almost all of the results have been on first order model theory, which studies mathematical structures that are axiomatized by first order theories (called elementary classes), e.g. group theory, Zermelo-Fraenkel set theory, Peano arithmetic, etc., and the most celebrated theorem has been Morley's categoricity theorem [Mor65].

On the other hand, other logics, usually more powerful than first order logic, have also been studied, among them infinitary logics: $\mathbb{L}_{\omega_1,\omega}$ where countable conjunctions and disjunctions are allowed, $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$, adding the quantifier "there exist uncountably many" to $\mathbb{L}_{\omega_1,\omega}$, and more generally $\mathbb{L}_{\kappa,\lambda}$ for larger cardinals $\lambda$ and $\kappa$ where conjunctions and disjunctions of $< \kappa$ formulas and quantification of $\lambda$ variables are allowed. See [Kei71] and [Dic75]. A major difficulty of working with infinitary logics is the absence of the compactness theorem. In fact, first order logic is the "strongest" one that has both the compactness theorem and the downward Löwenheim-Skolem-Tarski theorem [Lin69].

There are reasons why model theory should not be restricted to the study of elementary classes only despite the great success.

1. There are classes of structures that occur naturally in mathematics that are not elementary, such as periodic groups, locally finite groups, Noetherian rings, etc..

2. It is always nice to prove theorems in general contexts. The fact that certain results failing to generalize also provides insight of the field.
3. Within model theory, non-elementary classes occur as well. For example, the class of $\aleph_1$-saturated model of a countable first order $T$.

However, as discussed in [She09a, §N], "non-elementary classes" is not a well-defined notion to study. In order to prove meaningful results one has to make some restrictions. There are several candidates of such a framework, including excellent classes [She83a, She83b], universal classes [She87b] (now [She09a, §V]), homogeneous model theory [She71a, She75c] and abstract elementary classes, etc..

In the 1970s, Shelah [She87a] introduced the context of *abstract elementary classes* (AECs), an axiomatic framework that generalizes elementary classes and the infinitary logics listed like $\mathbb{L}_{\lambda^+,\omega}(\mathbf{Q})$. Typically, developments in this field have been guided by a few open problems and programs.

## 1.1 Dividing lines

In his book [She78], Shelah began the program of *classification theory*, that aims to divide first order theories into "simple" and "structural" ones versus "chaotic" and "unstructural" ones. A property (of a first order theory) that classifies first order theories in such a way is usually called a *dividing line*: the theories on the "simple" side should exhibit nice properties, such as a bound of the number of types, existence of a well-behaved notion of independence, etc., while the theories on the "chaotic" side should exhibit complex properties, such as having many models up to isomorphism (of a fixed cardinality), being able to encode subsets of a cardinal, etc.. There have been many successful such dividing lines for first order model theory: stablity, NIP, simplicity, superstability, $\mathrm{NTP}_2$, o-minimality. It is a reasonable quest to consider the paralell of this program: *classification for abstract elementary classes* [She09a, She09b].

Several successful dividing lines in first order model theory have parallels for abstract elementary classes. An appropriate generalization of stability for AECs was introduced in [She99] building on many previous papers including [She71c] and [GS]. In the last forty years starting with [GS86] much was discovered about analogues of superstability.

See [Vas16b], [GV17], [BGVV17] and [Leu23b] for some recent work. For analogues of simplicity, see [GMA21].

## 1.2 The categoricity conjecture

Motivated by and generalizing Morley's theorem, Shelah proved the categoricity theorem for uncountable theories in 1970s. Both theorems were successful in guiding the development of the field. The following conjecture (first appeared in print in [She83a]) for $\mathbb{L}_{\omega_1,\omega}$:

**Conjecture 4.** *Let $T$ be a countable theory in $\mathbb{L}_{\omega_1,\omega}$. If $T$ is categorical in some $\lambda \geq \beth_{\omega_1}$, then $T$ is categorical in all $\mu \geq \beth_{\omega_1}$.*

Generalizing this to abstract elementary classes, Shelah also conjectured the following (appeared in [She00]) known as the categoricity conjecture for abstract elementary classes:

**Conjecture 5.** *Let $\mathbf{K}$ be an abstract elementary class. If $\mathbf{K}$ is categorical in some $\lambda \geq \beth_{(2^{LS(\mathbf{K})})^+}$, then $K$ is categorical in all $\mu \geq \beth_{(2^{LS(\mathbf{K})})^+}$.*

There is a perhaps easier version called the eventual categoricity conjecture for abstract elementary classes proposed by Grossberg:

**Conjecture 6.** *For every cardinal $\mu$ there is $\mu_0$ such that for all abstract elementary classes $\mathbf{K}$ with $LS(\mathbf{K}) \leq \mu$, if $\mathbf{K}$ is categorical in some $\lambda_0 \geq \mu_0$, then $\mathbf{K}$ is categorical in all $\lambda \geq \mu$.*

While many special cases are verified (e.g. the case where $\mathbf{K}$ is universal by Vasey [Vas17a, Vas17b] and the case; See also [SV] ), all of the three categoricity conjectures remain open despite many approximations.

## 1.3 The successive categoricity question

Another test question is the succesive categoricity question. Baldwin asked the following question in [Fri75]:

**Question 7.** *Let $\psi$ be an $\mathbb{L}_{\omega_1,\omega}$-formula in a countable language. If the class of models of $\psi$ is categorical in $\aleph_0$ and $\aleph_1$, does $\psi$ have a model in $\aleph_2$?*

Shelah proved this [She87a]. Grossberg raised a parallel question for abstract elementary classes [She01, Problem (5), p. 34] that remains open:

**Question 8.** *Let* $\mathbf{K}$ *be an AEC and* $\lambda \geq LS(\mathbf{K})$ *be an infinite cardinal. If* $\mathbf{K}$ *is categorical in* $\lambda$ *and* $\lambda^+$*, must* $\mathbf{K}$ *have a model of cardinality* $\lambda^{++}$*?*

Shelah spent much effort obtaining partial approximations [She01] that induced many new concepts, among them $\lambda$-good frames [She09a, §III].

There are many partial answers to this question. [She87a, 3.7], [She01], [She09b, §VI.0.(2)], [She09a, §II.4.13.3], [JS13, 3.1.9], [Vas16a, 8.9], [Vas18b, 12.1], [SV18, 5.8], [MAV18, 3.3, 4.4], [MA20, 4.2], [Vas22, 1.6, 3.7, 5.4], [Leu23a, 4.9].

Many of these results require $\lambda = \aleph_0$ and $\lambda^+ = \aleph_1$. Most of them assume stability-like or superstability-like properties.

In this thesis, we provide approximations to the successive categoricity question in the third and the fourth chapters, which are the following two theorems respectively.

**Theorem 9.** *Suppose* $\lambda^+ < 2^{\aleph_0}$*. Assume* $\mathbf{K}$ *has amalgamation in* $\lambda$*, no maximal model in* $\lambda$*, and is stable for* $\lambda$*-algebraic types in* $\lambda$*. If* $\mathbf{K}$ *is* $(<\lambda^+, \lambda)$*-local, then* $\mathbf{K}$ *has a model of cardinality* $\lambda^{++}$*.*

**Theorem 10.** *Let* $\mathbf{K}$ *be an AEC and let* $\lambda \geq \mathrm{LS}(\mathbf{K})$*. If* $\mathbf{K}$ *has amalgamation, no maximal model and is stable in* $\lambda$*, and splitting is continuous in* $\lambda$*, then* $\mathbf{K}$ *has a model in* $\lambda^{++}$*.*

The first result assumes a weaker condition on types, that is, we only bound the number of $\lambda$-algebraic types rather than all types. The second result applies to arbitrarily large without assuming $\lambda^+ < 2^{\aleph_0)}$, not even $\lambda^+ < 2^\lambda$.

In the fourth chapter, we also provide certain machinery for solving the successive categoricity conjecture. In particular, we prove that stability in $\lambda$ is equivalent to existence in $\lambda^{++}$ under certain assumptions.

**Theorem 11.** *Suppose* $\lambda^+ < 2^\lambda < 2^{\lambda^+}$*. Assume* $\mathbf{K}$ *is categorical in* $\lambda$ *and* $\lambda^+$*,* $\mathbf{K}$ *is* $(<\lambda^+, \lambda)$*-local and* $\mathbf{K}$ *is almost stable in* $\lambda$*. The following are equivalent.*

1. $\mathbf{K}$ *has a model in* $\lambda^{++}$*.*
2. $\mathbf{K}$ *is stable in* $\lambda$*.*

We further exploit the idea used to show (2) $\Longrightarrow$ (1) and obtain categoricity on a tail.

**Theorem 12.** *Let* $\mathbf{K}$ *be an AEC with weak amalgamation and let* $\lambda \geq \mathrm{LS}(\mathbf{K})$ *be such that* $\mathbf{K}$ *is* $\lambda$*-tame. Assume* $\mathbf{K}$ *has amalgamation in* $\lambda$*,* $\mathbf{K}$ *is stable in* $\lambda$*, and splitting is continuous in* $\lambda$*. If* $\mathbf{K}$ *is categorical in* $\lambda$ *and* $\lambda^+$*, then* $\mathbf{K}$ *is categorical in all* $\mu \geq \lambda$*.*

In the fifth chapter we propose a parallel of NIP (depdendent theories) for abstract elementary classes. We show that it behaves in the expected way: its negation leads to being able to encode subsets.

**Theorem 13.** *Suppose* $\mathbf{K}$ *is* $(<\aleph_0)$*-tame,* $M \in \mathbf{K}$*,* $C \subseteq |M|$*,* $\lambda := |C| \geq \beth_3(LS(\mathbf{K}))$ *and* $(ded\ \lambda)^{2^{LS(\mathbf{K})}} = ded\ \lambda$*. Suppose* $|\mathbf{S}^1(C; M)| > ded\ \lambda$*. Then there is* $N \in \mathbf{K}$*,* $\langle \bar{a}_n \in {}^m|N| \mid n < \omega\rangle$ *and* $\phi$ *in the language of the Galois Morleyization of* $\mathbf{K}$ *such that for every* $w \subseteq \omega$ *there is* $b_w \in |N|$ *such that for all* $i < \omega$*,*

$$N \models \phi(\bar{a}_i, b_w) \iff i \in w.$$

We also compute the Hanf number to encode subsets as well: if $\mathbf{K}$ can encode subsets [1] of $\beth_{(2^{LS(\mathbf{K})})^+}$ then it can encode subsets of any cardinal.

**Theorem 14.** *If* $\mathbf{K}$ *can encode subsets of* $\mu := \beth_{(2^{LS(\mathbf{K})})^+}$*, then it can encode subsets of any cardinal. That is, if there are* $M \in \mathbf{K}$*,* $\{a_i \mid i < \mu\} \subseteq |M|$*,* $\{b_w \mid w \subseteq \mu\} \subseteq |M|$ *such that for all* $w \subseteq \mu$*,*

$$i \in w \iff \phi(a_i, b_w),$$

*then we can replace* $\mu$ *above by any cardinal.*

[1] See Chapter 5 for details

# Chapter 2

# Preliminaries

## 2.1 Basics and notation

**Definition 15.** *A* language*, also called a* similarity type *or* signature*, consists of three sets, the elements of which are called function symbols, relation symbols, and constants, and a number for each function symbol and for each relation symbol, called its arity.*

**Definition 16.** *For infinite cardinals* $\kappa \geq \lambda$*,* $\mathbb{L}_{\kappa,\lambda}$ *is the logic with* $(< \kappa)$*-ary conjunctions and disjunctions, existential and universal quantification of* $< \lambda$*-variables in addition to first order logic.*

**Notation 17.**

1. *For any structure* $M$ *in some language, we denote its universe by* $|M|$*, and its cardinality by* $\|M\|$*.*
2. *for* $M$ *and* $N$ *structures in the same language,* $M \subseteq N$ *means that* $M$ *is a substructure of* $N$

**Definition 18.** *For any class* $\mathbf{K}$ *of* $\tau$*-structures for some language* $\tau$ *and any (infinite) cardinal* $\lambda \geq |\tau|$*,*

1. *let* $\mathbb{I}(\mathbf{K}, \lambda)$ *denote the number of structures in* $\mathbf{K}$ *of cardinality* $\lambda$ *up to isomorphism;*
2. *we say* $\mathbf{K}$ *is categorical in* $\lambda$ *if* $\mathbb{I}(\mathbf{K}, \lambda) = 1$*.*

**Definition 19.** *An* abstract class *is* $(\mathbf{K}, \leq_{\mathbf{K}})$ *such that:*

1. $\mathbf{K}$ *is a class of* $\tau$*-structures for some finitary language* $\tau$*.*
2. $\leq$ *is a partial order on* $\mathbf{K}$*;*
3. *if* $M \leq_{\mathbf{K}} N$ *for some* $M$*,* $N \in \mathbf{K}$*, then* $M \subseteq N$*;*

4. $\leq_{\mathbf{K}}$ *respects isomorphisms:*
   (a) *If* $N \in \mathbf{K}$ *and* $M \cong N$, *then* $M \in \mathbf{K}$;
   (b) *If* $M \leq_{\mathbf{K}} N$ *and* $f : N \cong N'$, *then* $f[M] \leq_{\mathbf{K}} N'$. *That is,*

$$\begin{array}{ccc} M & \xrightarrow{id} & N \\ \downarrow f & & \downarrow f \\ f[M] & \overset{id}{\dashrightarrow} & N' \end{array}$$

   *commutes.*

**Definition 20.** *An* abstract elementary class *is an abstract class* $(\mathbf{K}, \leq_{\mathbf{K}})$ *satisfying additional properties:*

1. *coherence: If* $M_0 \leq_{\mathbf{K}} M_2$, $M_1 \leq_{\mathbf{K}} M_2$, $M_0 \subseteq M_1$, *then* $M_0 \leq_{\mathbf{K}} M_1$;
2. *Löwenheim-Skolem-Tarski axiom: the infinite cardinal* $LS(\mathbf{K})$ *is the smallest such that* $LS(\mathbf{K}) \geq |\tau|$, *and for all* $N \in \mathbf{K}$ *and* $A \subseteq |N|$, *there is* $M \leq_{\mathbf{K}} N$ *containing* $A$ *with* $\|M\| \leq_{\mathbf{K}} LS(\mathbf{K}) + |A|$;
3. *chain axioms: for an ordinal* $\alpha$, $\langle M_i : i < \alpha \rangle$ *such that* $i < j$ *implies* $M_i \leq_{\mathbf{K}} M_j$, $\bigcup_{i<\alpha} M_i \in \mathbf{K}$. *Moreover, if* $M_i \leq_{\mathbf{K}} N$ *for all* $i < \alpha$, *then* $\bigcup_{i<\alpha} M_i \leq_{\mathbf{K}} N$.

**Definition 21.** *For an abstract class* $\mathbf{K}$,

1. $\mathbf{K}$ *has the* amalgamation property *if for every* $M_0 \leq_{\mathbf{K}} M_l$ *for* $\ell = 1, 2$, *there is* $N \in \mathbf{K}$ *and* $\mathbf{K}$*-embeddings* $f_\ell : M_\ell \to N$ *for* $\ell = 1, 2$ *such that* $f_1 \restriction M_0 = f_2 \restriction M_0$;
2. $\mathbf{K}$ *has the* joint embedding property *if for every* $M_0$, $M_1 \in \mathbf{K}$ *there is* $N \in \mathbf{K}$ *such that* $M_0$ *and* $M_1$ $\mathbf{K}$*-embed into* $N$;
3. $\mathbf{K}$ *has* no maximal models *if every* $M \in \mathbf{K}$ *has a proper* $\leq_{\mathbf{K}}$*-extension in* $\mathbf{K}$.

**Notation 22.** *For a cardinal* $\lambda$, *we let* $\mathbf{K}_\lambda = \{M \in \mathbf{K} : \|M\| = \lambda\}$. *When we write* $M \leq_{\mathbf{K}} N$ *we assume that* $M, N \in \mathbf{K}$.

**Remark 23.** *For a property* $P$ *as in the previous definition, we say that* $\mathbf{K}$ *has* $P$ *in* $\lambda$ *if* $\mathbf{K}_\lambda$ *has the property* $P$.

## 2.2 Galois types

**Definition 24.** *Let* $\mathbf{K}$ *be an abstract elementary class.*

1. $\mathbf{K}^3 := \{(\bar{a}, A, N) \mid N \in \mathbf{K}, A \subseteq |N|, \bar{a}$ *is a sequence from* $|N|\}$.

2. *For* $(\bar{a}_0, A, N_0), (\bar{a}_1, A, N_1) \in \mathbf{K}^3$, $(\bar{a}_0, A, N_0)E_{at}(\bar{a}_1, A, N_1)$ *if there are* $N \in \mathbf{K}$, $f_0 : N_0 \to_A N$, *and* $f_1 : N_1 \to_A N$ $\mathbf{K}$*-embeddings such that* $f_0(\bar{a}_0) = f_1(\bar{a}_1)$, $f_0 \upharpoonright A = f_1 \upharpoonright A$.
3. $E$ *is the transitive closure of* $E_{at}$.
4. *For* $(\bar{a}, A, N) \in \mathbf{K}^3$, the Galois type of $\bar{a}$ over $A$ in $N$ *is* $\mathbf{gtp}(a/A, N) := [(a, A, N)]_E$.
5. *For* $N \in \mathbf{K}$ *and* $A \subseteq |N|$, $\alpha$ *an ordinal or* $\infty$, $\mathbf{S}^{<\alpha}(A; N) := \{\mathbf{gtp}(\bar{a}/A, N) \mid (\bar{a}, A, N) \in \mathbf{K}^3$ *and* $\bar{a} \in {}^{<\alpha}|N|\}$. $\mathbf{S}^{\alpha}(A; N)$ *is defined similarly. When* $\alpha = 1$ *we usually omit* $\alpha$.
6. *For* $M \in \mathbf{K}$, $\mathbf{S}^{\alpha}(M) := \bigcup_{N \geq_{\mathbf{K}} M} \mathbf{S}^{\alpha}(M; N)$.

**Remark 25.** *When* $\mathbf{K}$ *has the amalgamation property, then* $E_{at}$ *is already transitive, and hence* $E = E_{at}$.

**Definition 26.** *Let* $\mathbf{K}$ *be an abstract elementary class.*

1. *For* $N \in \mathbf{K}$, $q = \mathbf{gtp}(b/A, N) \in \mathbf{S}(A; N)$ *and* $B \subseteq A$, *let* $q \upharpoonright B := \mathbf{gtp}(b/B, N)$.
2. *In the situation above, let* $p \in \mathbf{S}(B; M)$ *for some* $M \in \mathbf{K}$, *we say* $p \leq q$ *if* $q \upharpoonright B = M$.

**Definition 27.** *Assume that* $\mathbf{K}_\lambda$ *has amalgamation. For* $M$, $N \in \mathbf{K}$, $p \in \mathbf{S}(M)$ *and* $\mathbf{K}$*-embedding* $h : M \to N$, *we define* $h(p) := \mathbf{gtp}(h'(a)/h[M], N)$, *where* $h' : M' \to N'$ *extends* $h$ *and* $(a, M, M') \in p$. *Note that* $h(p)$ *does not depend on the choice of* $(a, M, M')$ *or* $h'$. *See [Leu23b, 3.1] for a proof.*

**Definition 28.** *[She01, 0.22(2)] Let* $\mu > \lambda$. $N \in \mathbf{K}_\mu$ *is* saturated in $\mu$ above $\lambda$ *if for all* $M \leq_{\mathbf{K}} N$, $\lambda \leq \|M\| < \mu$, $N$ *realizes every type in* $\mathbf{S}(M)$.

**Definition 29.** *[She01, 0.26(1)] Let* $\mu > \lambda$. $N \in \mathbf{K}_\mu$ *is* homogeneous in $\mu$ for $\lambda$ *if for all* $M_1 \leq_{\mathbf{K}} N$, $M_1 \leq_{\mathbf{K}} M_2 \in \mathbf{K}_\lambda$, $\lambda \leq \|M_1\| \leq \|M_2\| < \mu$, *there is* $\mathbf{K}$*-embedding* $f : M_2 \to N$ *above* $M_1$.

**Fact 30.** *[She01, 0.26(1)] Let* $\mu > \lambda$. *If* $\mathbf{K}_\lambda$ *has amalgamation then* $M \in \mathbf{K}_\mu$ *is saturated over* $\mu$ *for* $\lambda$ *if and only if* $M$ *is homogeneous over* $\mu$ *for* $\lambda$.

In general an increasing sequence of Galois types do not have an upper bound, and the upper bound is not unique when there is one. However, a *coherent* sequence of Galois types does have an upper bound.

**Definition 31.** *Let* $\langle M_i : i < \delta \rangle$ *be an increasing continuous chain. A sequence of types* $\langle p_i \in \mathbf{S}(M_i) : i < \delta \rangle$ *is* coherent *if there are* $(a_i, N_i)$ *for* $i < \delta$ *and* $f_{j,i} : N_j \to N_i$ *for* $j < i < \delta$ *such that:*

1. $f_{k,i} = f_{j,i} \circ f_{k,j}$ *for all* $k < j < i$.
2. $\mathbf{gtp}(a_i/M_i, N_i) = p_i$.
3. $f_{j,i} \upharpoonright M_j = id_{M_j}$.
4. $f_{j,i}(a_j) = a_i$.

**Fact 32.** *Let* $\delta$ *be a limit ordinal and* $\langle M_i : i \leq \delta \rangle$ *be an increasing continuous chain. If* $\langle p_i \in \mathbf{S}^{na}(M_i) : i < \delta \rangle$ *is a coherent sequence of types, then there is* $p \in \mathbf{S}^{na}(M_\delta)$ *such that* $p \geq p_i$ *for every* $i < \delta$ *and* $\langle p_i \in \mathbf{S}^{na}(M_i) : i < \delta + 1 \rangle$ *is coherent.*

Even an increasing sequence of Galois types has an upper bound, it might not be necessarily unique. The following property called locality, which first appeared in print in [She01], asserts unique upper bounds for increasing sequence of Galois types.

**Definition 33.** 1. $\mathbf{K}$ *is* $(\kappa, \lambda)$-local *if for every* $M \in \mathbf{K}_\lambda$*, every increasing continuous chain* $\langle M_i : i < \kappa \rangle$ *such that* $M = \bigcup_{i<\kappa} M_i$ *and every* $p, q \in \mathbf{S}(M)$*, if* $p \upharpoonright M_i = q \upharpoonright M_i$ *for all* $i < \kappa$ *then* $p = q$.

2. $\mathbf{K}$ *is* $(< \kappa, \lambda)$-local *if* $\mathbf{K}$ *is* $(\mu, \lambda)$*-local for all* $\mu < \kappa$.

The notion of *tameness* is another such property that captures local behaviour of Galois types. Tameness appears in some of the arguments of [She99] and was isolated in [GV05].

**Definition 34.**

1. $\mathbf{K}$ *is* $(\kappa, \lambda)$-tame *if for every* $M \in \mathbf{K}_\lambda$ *and every* $p, q \in \mathbf{S}(M)$*, if* $p \neq q$*, then there is* $A \subseteq |M|$ *of cardinality* $\kappa$ *such that* $p \upharpoonright A \neq q \upharpoonright A$.
2. $\mathbf{K}$ *is* $(< \kappa, \lambda)$-tame *if for every* $M \in \mathbf{K}_\lambda$ *and every* $p, q \in \mathbf{S}(M)$*, if* $p \neq q$*, then there is* $A \subseteq |M|$ *of cardinality less than* $\kappa$ *such that* $p \upharpoonright A \neq q \upharpoonright A$.

Boney [Bon14] proved that if there is $\mu > LS(\mathbf{K})$ strongly compact then $\mathbf{K}$ is $\mu$-tame. Boney and Unger [BU17] proved that if every abstract elementary class is tame then there is an almost strongly compact cardinal. Boney, Kolesnikov, Grossberg and Vasey [BGKV16] derived tameness from model theoretic properties.

Below are some relations between tameness and locality.

**Proposition 35.** *Let* $\lambda \geq \mathrm{LS}(\mathbf{K})$.

1. *If* $\mathbf{K}$ *is* $(< \aleph_0, \lambda)$*-tame, then* $\mathbf{K}$ *is* $(< \lambda^+, \lambda)$*-local.*
2. *Assume* $\lambda > \mathrm{LS}(\mathbf{K})$*. If* $\mathbf{K}$ *is* $(\lambda, \lambda)$*-local, then* $\mathbf{K}$ *is* $(< \lambda, \lambda)$*-tame.*
3. *If* $\mathbf{K}$ *is* $(\mu, \mu)$*-local for every* $\mu \leq \lambda$*, then* $\mathbf{K}$ *is* $(\mathrm{LS}(\mathbf{K}), \mu)$*-tame for every* $\mu \leq \lambda$.

4. *Assume $\lambda \geq \kappa$, $cf(\kappa) > \chi$. If $\mathbf{K}$ is $(\chi, \lambda)$-tame, then $\mathbf{K}$ is $(\kappa, \lambda)$-local.*

*Proof.*

1. Straightforward.
2. Let $M \in \mathbf{K}_\lambda$ and $p, q \in \mathbf{S}(M)$ such that $p \restriction_A = q \restriction_A$ for every $A \subseteq |M|$ with $|A| < \lambda$. Let $\langle M_i : i < \lambda \rangle$ be an increasing continuous chain such that $M = \bigcup_{i<\lambda} M_i$ and $\|M_i\| \leq \mathrm{LS}(\mathbf{K}) + |i|$ for every $i < \lambda$. Since $\|M_i\| < \lambda$ for every $i < \lambda$, $p \restriction M_i = q \restriction M_i$ for every $i < \lambda$. Therefore, $p = q$ as $\mathbf{K}$ is $(\lambda, \lambda)$-local.
3. Similar to (2), see also [BL06, 1.18].
4. This is [BS08, 1.11]

□

**Remark 36.** *Boney proved that universal classes are $(< \aleph_0, \lambda)$-tame for every $\lambda \geq \mathrm{LS}(\mathbf{K})$ (see [Vas17a, 3.7]). Quasiminimal AECs are $(< \aleph_0, \lambda)$-tame for every $\lambda \geq \mathrm{LS}(\mathbf{K})$ [Vas18a, 4.18] and many natural AECs of module are $(< \aleph_0, \lambda)$-tame for every $\lambda \geq \mathrm{LS}(\mathbf{K})$ (see for example [MA23, §3]). The main results of Chapters 3 and 4 assume that the AEC is $(< \lambda^+, \lambda)$-local, so they apply to all of these classes.*

*On the other hand there are AECs which are not $(\aleph_1, \aleph_1)$-local [BS08] and which are not tame [BK09].*

Despite the importance of tameness in the development of AECs, the following question is still open.

**Question 37.** *If $\mathbf{K}$ is $(\aleph_0, \aleph_0)$-local, is $\mathbf{K}$ $(< \aleph_0, \aleph_0)$-tame?*

## 2.3 More on Galois types

In general a Galois type may not extend to all $\mathbf{K}$-extensions of its domain. That motivates the following definition.

**Definition 38.** *1. [She09b, §VI.1.8] $p = \mathbf{gtp}(a/M, N)$ has the $\lambda$-*extension property *if for every $M \leq_{\mathbf{K}} M' \in \mathbf{K}_\lambda$, there is $q \in \mathbf{S}^{na}(M')$ extending $p$. In this case we say $p \in \mathbf{S}^{\lambda-ext}(M)$.*[1]

[1]These types are also called *big types* in the literature, see for example [She75a] and [Les05].

2. *[She09b, §VI.1.15]* $p = \mathbf{gtp}(a/M, N)$ *is* $\lambda$-algebraic *if it has* $\leq \lambda$ *realizations in any* $\leq_{\mathbf{K}}$*-extension of* $M$*. In the case we denote* $p \in \mathbf{S}^{\lambda-al}(M)$*.*

By the following fact, it turns out that the $\lambda$-algebraic types are exactly the Galois types without the $\lambda$-extension property.

**Fact 39** ([She09a, VI.2.5(2B)])**.** *Assume* $\mathbf{K}$ *has amalgamation and no maximal model in* $\lambda$*.* $\mathbf{gtp}(a/M, N)$ *has* $\geq \lambda^+$ *realizations in some* $M' \in \mathbf{K}$ *such that* $M \leq_{\mathbf{K}} M'$ *if and only if* $\mathbf{gtp}(a/M, N)$ *has the* $\lambda$*-extension property.*

The following couple of notions appear in [She09a, §VI].

**Definition 40.** $\mathbf{S}_*$ *is* $\leq_{\mathbf{K}_\lambda}$-type-kind *when:*

1. $\mathbf{S}_*$ *is a function with domain* $\mathbf{K}_\lambda$*.*
2. $\mathbf{S}_*(M) \subseteq \mathbf{S}^{na}(M)$ *for every* $M \in \mathbf{K}_\lambda$*.*
3. $\mathbf{S}_*(M)$ *commutes with isomorphisms for every* $M \in \mathbf{K}_\lambda$*.*

**Definition 41.** $\mathbf{S}_1$ *is* hereditarily in $\mathbf{S}_2$ *when: for* $M \leq_{\mathbf{K}} N \in \mathbf{K}_\lambda$ *and* $p \in \mathbf{S}_2(N)$ *we have that if* $p \restriction M \in \mathbf{S}_1(M)$ *then* $p \in \mathbf{S}_1(N)$*. If* $\mathbf{S}_2 = \mathbf{S}^{na}$ *we will say that* $\mathbf{S}_1$ *is* hereditary*.*

The proof of following proposition is straightforward.

**Proposition 42.** *Assume* $\mathbf{K}$ *has amalgamation and no maximal model in* $\lambda$*.* $\mathbf{S}^{\lambda-al}$ *is* $\leq_{\mathbf{K}_\lambda}$*-type-kind and hereditary.*

**Definition 43.** *For* $M \in \mathbf{K}$ *and* $\Gamma \subseteq \mathbf{S}^{na}(M)$*,* $\Gamma$ *is* $\mathbf{S}_*$-inevitable *if for every* $N >_{\mathbf{K}} M$*, if there is* $p \in \mathbf{S}_*(M)$ *realized in* $N$ *then there is* $q \in \Gamma$ *realized in* $N$*.*

The following result appears in [She09a] without a proof. See the last paragraph of the introduction of Chapter 3 and Remark 65 for details.

**Fact 44** ([She09a, VI.2.11.(2)])**.** *Assume* $\mathbf{K}$ *has amalgamation and no maximal model in* $\lambda$*. If*

1. $\mathbf{S}_*$ *is* $\leq_{\mathbf{K}_\lambda}$*-type-kind and hereditary, and*
2. *for every* $N \in \mathbf{K}_\lambda$ *there is an* $\mathbf{S}_*$*-inevitable* $\Gamma_N \subseteq \mathbf{S}^{na}(N)$ *of cardinality* $\leq \lambda$*,*

*then for every* $M \in \mathbf{K}_\lambda$ *we have that* $|\mathbf{S}_*(M)| \leq \lambda$*.*

We recall one last definition from [She09a, §VI].

**Definition 45.** *Let* $M \in \mathbf{K}_\lambda$*.* $p \in \mathbf{S}^{na}(M)$ *is a* $< \lambda^+$-minimal type *if for every* $M \leq_{\mathbf{K}} N \in \mathbf{K}_\lambda$*,* $|\{q \in \mathbf{S}^{na}(N) : q \restriction M = p\}| \leq \lambda$*. Let* $\mathbf{S}^{<\lambda^+-min}(M)$ *denote the* $< \lambda^+$*-minimal types over* $M$*.*

## 2.4 The weak diamond and related results

The following principle known as the weak diamond was introduced by Devlin and Shelah [DS78].

**Definition 46.** *Let $S \subseteq \lambda^+$ be a stationary set. $\Phi^k_{\lambda^+}(S)$ holds if and only if for all $F : {}^{<\lambda^+}(2^\lambda) \to k$ there exists $g : \lambda^+ \to k$ such that for all $f : \lambda^+ \to 2^\lambda$ the set $\{\alpha \in S : F(f \restriction \alpha) = g(\alpha)\}$ is stationary. When $S = \lambda^+$ we write $\Phi^k_{\lambda^+}$ for $\Phi^k_{\lambda^+}(S)$.*

Proofs of the following facts can be consulted in [Gro2X, §15].

**Fact 47.** *Let $S \subseteq \lambda^+$ be a stationary set and $k < \omega$. If $\Phi^k_{\lambda^+}(S)$ holds, then for all*

$$F : \underbrace{{}^{<\lambda^+}(2^\lambda) \times \ldots \times {}^{<\lambda^+}(2^\lambda)}_{n \text{ times}} \to k$$

*there is $g : \lambda^+ \to k$ such that for all $f_i : \lambda^+ \to 2^\lambda$ for $i < n$ the set*

$$\{\alpha \in S : F(f_1 \restriction \alpha, \ldots, f_{n-1} \restriction \alpha) = g(\alpha)\}$$

*is stationary.*

**Fact 48.**

1. *$2^\lambda < 2^{\lambda^+}$ if and only if $\Phi^2_{\lambda^+}(\lambda^+)$ holds.*
2. *Suppose that $\Phi^2_{\lambda^+}$ holds. Then there are disjoint stationary sets $S_\alpha$ for $\alpha < \lambda^+$ such that $\Phi^2_{\lambda^+}(S_\alpha)$ holds for all $\alpha < \lambda^+$.*

**Fact 49.** *[She87a, 3.5] ($2^\lambda < 2^{\lambda^+}$) If $\mathbb{I}(\mathbf{K}, \lambda) = 1 \leq I(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$, then $\mathbf{K}_\lambda$ has amalgamation.*

# Chapter 3

# On the successive categoricity question

## 3.1 Introduction

In this chapter we present results of [MAY24] (joint work with Marcos Mazari-Armida, to appear in the Journal of Symbolic Logic). Recall Grossberg's succesive categoricity question:

**Question 50.** *Let* $\mathbf{K}$ *be an AEC and* $\lambda \geq LS(\mathbf{K})$ *be an infinite cardinal. If* $\mathbf{K}$ *is categorical in* $\lambda$ *and* $\lambda^+$*, must* $\mathbf{K}$ *have a model of cardinality* $\lambda^{++}$*?*

We provide a partial answer to the question assuming locality of Galois types and certain cardinal arithmetic. These cases are new and Shelah's original technology could not reach them.

**Theorem 51.** *Suppose* $\lambda^+ < 2^{\aleph_0}$*. Assume* $\mathbf{K}$ *has amalgamation in* $\lambda$*, no maximal model in* $\lambda$*, and is stable for* $\lambda$*-algebraic types in* $\lambda$*. If* $\mathbf{K}$ *is* $(<\lambda^+, \lambda)$*-local, then* $\mathbf{K}$ *has a model of cardinality* $\lambda^{++}$*.*

To help us compare the main theorem of this chapter to the results mentioned in the previous paragraph, let us recall the following three frameworks: universal classes [Tar54], [She87b], tame AECs [GV05] and local AECs [She01][BL06]. The first is a semantic assumption on the AEC while the other two are locality assumptions on Galois types (see Definition 34 and Definition 33). The relation between these frameworks is as follows:

universal classes are $(< \aleph_0, \lambda)$-tame for every $\lambda \geq \mathrm{LS}(\mathbf{K})$ [Vas17a, 3.7] and $(< \aleph_0, \lambda)$-tame AECs are $(< \lambda^+, \lambda)$-local for every $\lambda \geq \mathrm{LS}(\mathbf{K})$. The first inclusion is proper and the second inclusion is not known to be proper (see Question 37).

We do not assume that the AEC has a countable Löwenheim-Skolem-Tarski number or that $\lambda = \aleph_0$. When the AEC has a countable Löwenheim-Skolem-Tarski number and $\lambda = \aleph_0$, Theorem 51 is known for universal classes [MAV18, 3.3] but it is new for $(< \aleph_0, \aleph_0)$-tame AECs. Moreover, for $\lambda > \aleph_1$, Theorem 51 is new even for universal classes. Both [MAV18, 3.3, 4.4] and [Vas22, 1.6] assume that $2^{\aleph_0} < 2^{\aleph_1}$, but the assumption that $\lambda^+ < 2^{\aleph_0}$ is new although this is a weak assumption as long as $\lambda$ is a *small* cardinal.

When the AEC has a countable Löwenheim-Skolem-Tarski number and $\lambda = \aleph_0$, Theorem 51 for $(< \aleph_0, \aleph_0)$-tame AECs can be obtained using [SV18, 4.7], [SV18, 5.8], [She09a, II.4.13][1], but the result has never been stated in the literature. The argument presented in this paper is significantly simpler than the argument using the results of Shelah and Vasey. Moreover, the result is new for $(\aleph_0, \aleph_0)$-local AECs. Furthermore, for $\lambda > \aleph_1$, Theorem 62 is even new for universal classes.

Another result similar to Theorem 62 is [Vas18b, 12.1]. The main difference is that Vasey's result has the additional assumption that the AEC is categorical in $\lambda$. Moreover, Vasey assumes tameness while we only assume the weaker property of locality for Galois types. It is worth mentioning that Vasey does not assume that $\lambda < 2^{\aleph_0}$, but this is a weak assumption as long as $\lambda$ is a *small* cardinal.

The set theoretic assumption that $\lambda^+ < 2^{\aleph_0}$ can be replaced by the model theoretic assumption that $|\mathbf{S}^{na}(M)| < 2^{\aleph_0}$ for every $M \in \mathbf{K}_\lambda$ (see Theorem 61). When $2^{\aleph_0} > \lambda$ this assumption is weaker than stability in $\lambda$. When the AEC has a countable Löwenheim-Skolem-Tarski number and $\lambda = \aleph_0$, Theorem 61 is known for arbitrary AECs if $2^{\aleph_0} = \aleph_1$ and it is new for $(< \aleph_0, \aleph_0)$-tame AECs if $2^{\aleph_0} > \aleph_1$.

Most of the partial results towards an answer to Grossberg's question build a frame in the sense of [She09a, §II], our approach is different. Instead of building a global notion, we focus on finding one *good* type called a $\lambda$-unique type (see Lemma 55 and Lemma 59). Then we carefully build a chain of types above this type to show that every model of cardinality

[1]We were unaware of this argument until Sebastien Vasey pointed it out when we showed him a final draft of the paper.

$\lambda^+$ has a proper extension and hence show the existence of a model of cardinality $\lambda^{++}$. This makes our argument shorter than any of the previous arguments used to build larger models.

The results of the last section of this chapter rely on a result of Shelah [She09b, VI.2.11.(2)] (Fact 44 of this thesis) for which Shelah does not provide an argument and which we were unable to verify. Even if Shelah's result turns out to be false, Theorem 51 still holds if we assume that $\lambda < 2^{\aleph_0}$ and stability in $\lambda$ instead of assuming that $\lambda^+ < 2^{\aleph_0}$. For AECs with countable Löwenheim-Skolem-Tarski number and $\lambda = \aleph_0$, this weaker result is known for arbitrary AECs (see Remark 63), but the argument presented here is simpler than the previously known argument for $(< \aleph_1, \aleph_0)$-tame AECs as we do not need to construct a good $\aleph_0$-frame. The result does not assume stability compared to [Vas18b, 12.1], and it is new for $(< \lambda^+, \lambda)$-local AECs. See Remark 65 for more details.

## 3.2 Main results

**Definition 52.**

- $p = \mathbf{gtp}(a/M, N)$ *has the* $\lambda$-extension property *if for every* $M \leq_{\mathbf{K}} M' \in \mathbf{K}_\lambda$*, there is* $q \in \mathbf{S}^{na}(M')$ *extending* $p$*. In this case we say* $p \in \mathbf{S}^{\lambda-ext}(M)$*.*[2]
- $p = \mathbf{gtp}(a/M, N)$ *is* $\lambda$-algebraic *if* $p \in \mathbf{S}^{na}(M) - \mathbf{S}^{\lambda-ext}(M)$*. Let* $\mathbf{S}^{\lambda-al}(M)$ *denote the* $\lambda$*-algebraic types over* $M$*.*

Recall that if $p$ has the $\lambda$-extension property then $p$ is non-algebraic.

Recall that an AEC $\mathbf{K}$ is *stable in* $\lambda$ if $|\mathbf{S}(M)| \leq \lambda$ for every $M \in \mathbf{K}_\lambda$. We introduce a weakening of stability.

**Definition 53.** $\mathbf{K}$ *is stable for* $\lambda$*-algebraic types in* $\lambda$ *if for all* $M \in \mathbf{K}_\lambda$*,* $|\mathbf{S}^{\lambda-al}(M)| \leq \lambda$*.*

**Remark 54.** *Stability for* $\lambda$*-algebraic types in* $\lambda$ *is strictly weaker than stability in* $\lambda$*. Consider the case where* $\mathbf{K}$ *is an elementary class which is unstable in* $\lambda$*. In that case, all non-algebraic types have the extension property. Thus* $\mathbf{S}^{\lambda-al}(M) = \emptyset$ *for all* $M \in \mathbf{K}_\lambda$*, but for some* $M \in \mathbf{K}_\lambda$*,* $|\mathbf{S}(M)| > \lambda$*.*

We show that there are types with the $\lambda$-extension property.

[2] These types are also called *big types* in the literature, see for example [She75b] and [Les05].

**Lemma 55.** *Assume that* $\mathbf{K}$ *has amalgamation in* $\lambda$ *and no maximal model in* $\lambda$*. If* $\mathbf{K}$ *is stable for* $\lambda$*-algebraic types in* $\lambda$*, then there is* $p \in \mathbf{S}^{\lambda-ext}(M)$ *for every* $M \in \mathbf{K}_\lambda$*.*

*Proof.* Fix $M \in \mathbf{K}_\lambda$. There are two cases to consider:

Case 1: $|\mathbf{S}^{na}(M)| \geq \lambda^+$. This follows directly from the assumption that $\mathbf{K}$ is stable for $\lambda$-algebraic types in $\lambda$.

Case 2: $|\mathbf{S}^{na}(M)| \leq \lambda$. Since $\mathbf{K}$ has no maximal model in $\lambda$, there is $M \leq_{\mathbf{K}} N \in \mathbf{K}_{\lambda^+}$. Let $\Phi : |N| \backslash |M| \to \mathbf{S}^{na}(M)$ be given by $a \mapsto \mathbf{gtp}(a/M, N)$. Since $\|N| \backslash |M\| = \lambda^+$ and $|\mathbf{S}^{na}(M)| \leq \lambda$, by the pigeonhole principle there is $q \in \mathbf{S}^{na}(M)$ such that $|\{a \in |N| \backslash |M| : \Phi(a) = q\}| \geq \lambda^+$. That is, $q$ has $\lambda^+$-many realizations in $N$. Hence $q$ has the the $\lambda$-extension property by Fact 39. □

We will use the following strengthening of the extension property.

**Definition 56.** $p = \mathbf{gtp}(a/M, N)$ *has the* $\lambda$-strong extension property *if for every* $M \leq_{\mathbf{K}} M' \in \mathbf{K}_\lambda$*, there is* $q \in \mathbf{S}^{\lambda-ext}(M')$*.*

We show that the strong extension property is the same as the extension property if $\mathbf{K}$ is stable for $\lambda$-algebraic types.

**Lemma 57.** *Assume that* $\mathbf{K}$ *has amalgamation in* $\lambda$*, no maximal model in* $\lambda$ *and is stable for* $\lambda$*-algebraic types in* $\lambda$*. Let* $M \in \mathbf{K}_\lambda$ *and* $p \in \mathbf{S}^{na}(M)$*,* $p \in \mathbf{S}^{\lambda-ext}(M)$ *if and only* $p$ *has the* $\lambda$*-strong extension property.*

*Proof.* We only need to show the forward direction. Let $N \geq_{\mathbf{K}} M$ and $\{a_i \in |N| : i < \lambda^+\}$ realizing $p$. Let $M \leq_{\mathbf{K}} M^* \in \mathbf{K}_\lambda$. Using amalgamation in $\lambda$ we may assume that $M^* \leq_{\mathbf{K}} N$. Moreover, we may assume without loss of generality that for all $i < \lambda^+$, $a_i \notin |M^*|$. If not, subtract those $a_i$ that are in $M^*$. Observe that $\mathbf{gtp}(a_i/M^*, N) \geq p$ for all $i < \lambda^+$. If $|\{\mathbf{gtp}(a_i/M^*, N) : i < \lambda^+\}| = \lambda^+$, we are done by stability for $\lambda$-algebraic types. Otherwise $|\{\mathbf{gtp}(a_i/M^*, N) : i < \lambda^+\}| \leq \lambda$. Then a similar argument to that of Case 2 of the previous lemma can be used to obtain result. □

Recall the following notion. This notion was first introduced by Shelah in [She75b, 6.1], called minimal types there. Note that this is a different notion from the minimal types of [She01]. These types are also called *quasiminimal types* in the literature, see for example [Zil05] and [Les05].

**Definition 58.** $p = \mathbf{gtp}(a/M, N)$ *is a* $\lambda$-unique type *if*

1. *$p = \mathbf{gtp}(a/M, N)$ has the $\lambda$-extension property.*
2. *For every $M \leq_{\mathbf{K}} M' \in \mathbf{K}_\lambda$, $p$ has at most one extension $q \in \mathbf{S}^{\lambda-ext}(M')$.*

*In this case we say that $p \in \mathbf{S}^{\lambda-unq}(M)$.*

We show the existence of $\lambda$-unique types.

**Lemma 59.** *Assume that $\mathbf{K}$ has amalgamation in $\lambda$, no maximal model in $\lambda$ and is stable for $\lambda$-algebraic types in $\lambda$. If $|\mathbf{S}^{na}(M)| < 2^{\aleph_0}$ for every $M \in \mathbf{K}_\lambda$, then for every $M_0 \in \mathbf{K}_\lambda$ and $p \in \mathbf{S}^{\lambda-ext}(M_0)$, there is $M_1 \in \mathbf{K}_\lambda$ and $q \in \mathbf{S}^{\lambda-unq}(M_1)$ such that $M_0 \leq_{\mathbf{K}} M_1$ and $q$ extends $p$.*

*Proof.* Assume for the sake of contradiction that this is not the case. Let $M_0 \in \mathbf{K}_\lambda$ and $p \in \mathbf{S}^{\lambda-ext}(M_0)$ without a $\lambda$-unique type above it.

We build $\langle M_n : n < \omega \rangle$ and $\langle p_\eta : \eta \in 2^{<\omega} \rangle$ by induction such that:

1. $p_{\langle\rangle} = p$;
2. for every $\eta \in 2^{<\omega}$, $p_\eta \in \mathbf{S}^{\lambda-ext}(M_{\ell(\eta)})$;
3. for every $\eta \in 2^{<\omega}$, $p_{\eta^\frown 0} \neq p_{\eta^\frown 1}$.

Construction: The base step is given so we do the induction step. By induction hypothesis we have $\langle p_\eta \in \mathbf{S}^{\lambda-ext}(M_n) : \eta \in 2^n \rangle$. Since there is no $\lambda$-unique type above $p_{\langle\rangle}$ and by Lemma 57, for every $\eta \in 2^n$ there are $N_\eta \in \mathbf{K}_\lambda$ and $q^0_\eta$, $q^1_\eta \in \mathbf{S}^{\lambda-ext}(N_\eta)$ such that $q^0_\eta$, $q^1_\eta \geq p_\eta$ and $q^0_\eta \neq q^1_\eta$.

Using amalgamation in $\lambda$ we build $M_{n+1} \in \mathbf{K}_\lambda$ and $\langle f_\eta : N_\eta \xrightarrow[M_n]{} M_{n+1} : \eta \in 2^n \rangle$. Now for every $\eta \in 2^n$, let $p_{\eta^\frown 0}, p_{\eta^\frown 1} \in \mathbf{S}^{\lambda-ext}(M_{n+1})$ such that $p_{\eta^\frown 0} \geq f_\eta(q^0_\eta)$ and $p_{\eta^\frown 1} \geq f_\eta(q^1_\eta)$. These exist by Lemma 57. It is easy to show that $M_{n+1}$ and $\langle p_{\eta^\frown \ell} : \eta \in 2^n, \ell \in \{0, 1\} \rangle$ are as required.

Enough: Let $N := \bigcup_{n<\omega} M_n \in \mathbf{K}_\lambda$. For every $\eta \in 2^\omega$, let $p_\eta \in \mathbf{S}^{na}(N)$ be an upper bound of $\langle p_{\eta\restriction_n} : n < \omega \rangle$ given by Fact 32. Observe that if $\eta \neq \nu \in 2^\omega$, $p_\eta \neq p_\nu$. Indeed, let $n$ be the minimum $n$ such that $\eta \restriction_n = \nu \restriction_n$ and $\eta(n) \neq \nu(n)$. Then $p_\eta \restriction_{M_{n+1}} = p_{\eta\restriction_n{}^\frown \eta(n)} \neq p_{\nu\restriction_n{}^\frown \nu(n)} = p_\nu \restriction_{M_{n+1}}$ by Condition (3) of the construction. Then $|\mathbf{S}^{na}(N)| \geq 2^{\aleph_0}$ which contradicts our assumption. $\square$

**Remark 60.** *If $M \leq_{\mathbf{K}} N$, $p \in \mathbf{S}^{\lambda-unq}(M)$, $q \in \mathbf{S}^{\lambda-ext}(N)$ and $q \geq p$, then $q \in \mathbf{S}^{\lambda-unq}(N)$.*

We are ready to prove one of the main results of the paper.

**Theorem 61.** *Assume that* $\mathbf{K}$ *has amalgamation in* $\lambda$*, no maximal model in* $\lambda$*, and is stable for* $\lambda$*-algebraic types in* $\lambda$*. If* $|\mathbf{S}^{na}(M)| < 2^{\aleph_0}$ *for every* $M \in \mathbf{K}_\lambda$ *and* $\mathbf{K}$ *is* $(< \lambda^+, \lambda)$*-local, then* $\mathbf{K}$ *has a model of cardinality* $\lambda^{++}$*.*

*Proof.* It is enough to show that $\mathbf{K}$ has no maximal models in $\lambda^+$.

Assume for the sake of contradiction that $M \in \mathbf{K}_{\lambda^+}$ is a maximal model. Let $N \leq_{\mathbf{K}} M$ such that $N \in \mathbf{K}_\lambda$. By the maximality of $M$ together with Lemma 55, Lemma 59 and amalgamation in $\lambda$, there is $M_0 \in \mathbf{K}_\lambda$ with $N \leq_{\mathbf{K}} M_0 \leq_{\mathbf{K}} M$ and $q_0 \in \mathbf{S}^{\lambda-unq}(M_0)$. Let $\langle M_i \in \mathbf{K}_\lambda : i < \lambda^+\rangle$ be a resolution of $M$ with $M_0$ as before. We build $\langle p_i : i < \lambda^+\rangle$ such that:

1. $p_0 = q_0$;
2. if $i < j < \lambda^+$, then $p_i \leq p_j$;
3. for every $i < \lambda^+$, $p_i \in \mathbf{S}^{\lambda-unq}(M_i)$;
4. for every $j < \lambda^+$, $\langle p_i : i < j\rangle$ is coherent .

Construction The base step is given and the successor step can be achieved using Lemma 57 and Remark 60. So assume $i$ is limit, take $p_i$ to be an upper bound of $\langle p_j : j < i\rangle$ given by Fact 32. By Fact 32 $\langle p_j : j < i + 1\rangle$ is coherent so we only need to show that $p_i \in \mathbf{S}^{\lambda-unq}(\bigcup_{j<i} M_j)$.

By Remark 60 it suffices to show that $p_i \in \mathbf{S}^{\lambda-ext}(\bigcup_{j<i} M_j)$. Since $p_0 \in \mathbf{S}^{\lambda-unq}(M_0)$ and $M_0 \leq_{\mathbf{K}} \bigcup_{j<i} M_j$, there is $q \in \mathbf{S}^{\lambda-ext}(\bigcup_{j<i} M_j)$ such that $q \geq p_0$ by Lemma 57.

We show that for every $j < i$, $q \upharpoonright_{M_j} = p_i \upharpoonright_{M_j}$. Let $j < i$. Since $q \upharpoonright_{M_j} \in \mathbf{S}^{\lambda-ext}(M_j)$, $p_i \upharpoonright_{M_j} = p_j \in \mathbf{S}^{\lambda-ext}(M_j)$ and both extend $p_0$ a $\lambda$-unique type, $q \upharpoonright_{M_j} = p_i \upharpoonright_{M_j}$.

Therefore, $q = p_i$ as $\mathbf{K}$ is $(< \lambda^+, \lambda)$-local. Hence $p_i \in \mathbf{S}^{\lambda-ext}(\bigcup_{j<i} M_j)$ as $q \in \mathbf{S}^{\lambda-ext}(\bigcup_{j<i} M_j)$.

Enough Let $q^* \in \mathbf{S}^{na}(M)$ be an upper bound of the coherent sequence $\langle p_i : i < \lambda^+\rangle$ given by Fact 32. As $q^*$ is a non-algebraic type, $M$ has a proper extension which contradicts our assumption that $M$ is maximal. □

We use the previous theorem to obtain two corollaries with more natural assumptions. The next result is the result mentioned in the abstract.

**Theorem 62.** *Suppose $\lambda < 2^{\aleph_0}$. Let $\mathbf{K}$ be an abstract elementary class with $\lambda \geq \mathrm{LS}(\mathbf{K})$. Assume $\mathbf{K}$ has amalgamation in $\lambda$, no maximal model in $\lambda$, and is stable in $\lambda$. If $\mathbf{K}$ is $(< \lambda^+, \lambda)$-local, then $\mathbf{K}$ has a model of cardinality $\lambda^{++}$.*

*Proof.* We show that for every $M \in \mathbf{K}_\lambda$, $|\mathbf{S}^{na}(M)| < 2^{\aleph_0}$. This is enough by Theorem 61. Let $M \in \mathbf{K}_\lambda$. $|\mathbf{S}^{na}(M)| \leq \lambda$ by stability in $\lambda$. Since $\lambda < 2^{\aleph_0}$ by assumption, $|\mathbf{S}^{na}(M)| < 2^{\aleph_0}$. □

**Remark 63.** *For AECs $\mathbf{K}$ with $\mathrm{LS}(\mathbf{K}) = \aleph_0$ and $\lambda = \aleph_0$, the assumption that $\lambda < 2^{\aleph_0}$ is vacuous. This result for $(< \aleph_0, \aleph_0)$-tame AECs can be obtained using [SV18, 4.7], [SV18, 5.8], [She09a, II.4.13], but the result has never been stated in the literature. Moreover, the argument presented in this paper is significantly simpler than the argument using the results of Shelah and Vasey. Furthermore, the result is new for $(\aleph_0, \aleph_0)$-local AECs.*

We can also weaken the stability assumption to stability for $\lambda$-algebraic types at the cost of strengthening the cardinal arithmetic hypothesis from $\lambda < 2^{\aleph_0}$ to $\lambda^+ < 2^{\aleph_0}$.

**Theorem 64.** *Suppose $\lambda^+ < 2^{\aleph_0}$. Assume $\mathbf{K}$ has amalgamation in $\lambda$, no maximal model in $\lambda$, and is stable for $\lambda$-algebraic types in $\lambda$. If $\mathbf{K}$ is $(< \lambda^+, \lambda)$-local, then $\mathbf{K}$ has a model of cardinality $\lambda^{++}$.*

*Proof.* Assume for the sake of contradiction that $\mathbf{K}_{\lambda^{++}} = \emptyset$. We show that for every $M \in \mathbf{K}_\lambda$, $|\mathbf{S}^{na}(M)| < 2^{\aleph_0}$. This is enough by Theorem 61.

Let $M \in \mathbf{K}_\lambda$. Then there is $M \leq_{\mathbf{K}} N \in \mathbf{K}_{\lambda^+}$ maximal. Every $p \in \mathbf{S}^{na}(M)$ is realized in $N$ by amalgamation in $\lambda$ and maximality of $N$. Thus $|\mathbf{S}^{na}(M)| \leq \|N\| = \lambda^+$. Since $\lambda^+ < 2^{\aleph_0}$ by assumption, $|\mathbf{S}^{na}(M)| < 2^{\aleph_0}$. □

## 3.3 Additional results

In this section we present a natural assumption under which an AEC is stable for $\lambda$-algebraic types. We use this result (69) together with the results of the previous section to give a positive answer to Grossberg's question for small cardinals assuming a mild locality condition for Galois types and without any stability assumptions. *All* the main results of this section rely on a result of Shelah [She09a, VI.2.11.(2)] (Fact 44 of this paper) for which

Shelah does not provide an argument, for which the *standard* argument does not seem to work, and which we were unable to verify. See Remark 65 for more details.

**Remark 65.** *As mentioned in the introduction, the* standard *argument does not seem to work and we were unable to verify Fact 44. The standard argument we are referencing here is the one used to show stability from the existence of a good* $\lambda$*-frame [She09a, II.4.2] . The reason that argument does not work is because we do not have any trace of local character. It is worth mentioning that the following two generalizations [JS13, 2.5.8] and [Vas20, A.11] of that argument do not work either.*

*We give additional details on why the standard argument does not work, hoping that this could help elucidate the situation and eventually help prove the result. In the standard argument when* $\mathbf{S}_* = \mathbf{S}^{na}$*, one builds* $\langle M_i \in K_\lambda : i \leq \lambda \rangle$ *increasing and continuous with each* $M_{i+1}$ *realizing* $\Gamma_{M_i}$*, hoping that eventually we realize all types in* $\mathbf{S}_*(M_0)$ *in* $M_\lambda$*. To show any type* $\mathbf{gtp}(a/M_0, N) \in \mathbf{S}_*(M_0)$ *is realized, one builds* $\langle N_i : i < \lambda^+ \rangle$ *and* $\mathbf{K}$*-embeddings* $f_i : M_i \to N_i$ *and shows that* $f_\lambda$ *is an isomorphism, and hence* $f_\lambda^{-1} \restriction_{N_0}: N_0 \to M_\lambda$ *is enough. If* $f_\lambda$ *is not an isomorphism, some type in* $\mathbf{S}^{na}(M_\lambda)$ *and hence some type in* $\Gamma_{M_\lambda}$ *is realized in* $N_\lambda$*, and using local character, that realization can be "resolved" at some stage* $i < \lambda$*. Adapting this naively to the case when* $\mathbf{S}_*$ *is not necessarily* $\mathbf{S}^{na}$*, we expect that no type in* $\Gamma_{M_\lambda}$ *and hence no type in* $\mathbf{S}_*(M_\lambda)$ *is realized in* $N_\lambda$ *via* $f_\lambda$*. Without local character we cannot realize* $\Gamma_{M_\lambda}$ *in earlier stages. However this is more than what is needed and might be unnecessary, as this would imply that no* $\mathbf{S}_*(M_\lambda)$ *is realized in* $N_\lambda$*, while we only need that there is no extension of any type in* $\mathbf{S}_*(M_0)$ *to* $\mathbf{S}^{na}(M_\lambda)$ *(the type is in* $\mathbf{S}_*(M_\lambda)$ *since* $\mathbf{S}_*$ *is hereditary) is realized in* $|N_\lambda| - |M_\lambda|$*. Also, for our purpose, it would be enough if Fact 44 could be proved under the assumptions of Theorem 69.*

We recall one last definition from [She09a, §VI].

**Definition 66.** *Let* $M \in \mathbf{K}_\lambda$*.* $p \in \mathbf{S}^{na}(M)$ *is a* $< \lambda^+$-minimal type *if for every* $M \leq_{\mathbf{K}} N \in \mathbf{K}_\lambda$*,* $|\{q \in \mathbf{S}^{na}(N) : q \restriction_M = p\}| \leq \lambda$*. Let* $\mathbf{S}^{<\lambda^+ - min}(M)$ *denote the* $< \lambda^+$*-minimal types over* $M$*.*

**Lemma 67.** *Assume* $\mathbf{K}$ *has amalgamation in* $\lambda$*. For every* $M \in \mathbf{K}_\lambda$*,* $\mathbf{S}^{\lambda - al}(M) \subseteq \mathbf{S}^{<\lambda^+ - min}(M)$*.*

*Proof.* Fix $M \in \mathbf{K}_\lambda$. We show the result by contrapositive. Let $p \in \mathbf{S}^{na}(M) - \mathbf{S}^{<\lambda^+ - min}(M)$, i.e., $p$ has at least $\lambda^+$ extensions to $\mathbf{S}^{na}(N)$ for some $M \leq_{\mathbf{K}} N \in \mathbf{K}_\lambda$. Using the amalgamation property in $\lambda$ one can construct $M^* \in \mathbf{K}_{\lambda^+}$ such that $M \leq_{\mathbf{K}} N \leq_{\mathbf{K}} M^*$ and $M^*$ realizes $\lambda^+$ many extensions of $p$ to $\mathbf{S}^{na}(N)$. In particular, $M^*$ has $\lambda^+$ realizations of $p$. Hence $p$ has the $\lambda$-extension property by Fact 39. $\square$

**Fact 68** ([She09a, VI.2.18])**.** *($2^\lambda < 2^{\lambda^+}$) Assume $\mathbf{K}$ has amalgamation and no maximal model in $\lambda$. If*

1. *$\mathbf{S}_*$ is $\leq_{\mathbf{K}_\lambda}$-type-kind and hereditary,*
2. *$\mathbf{S}_* \subseteq \mathbf{S}^{<\lambda^+ - min}$, and*
3. *there is $M \in \mathbf{K}_\lambda$ such that:*
   - *(a) $|\mathbf{S}_*(M)| \geq \lambda^+$, and*
   - *(b) if $M \leq_{\mathbf{K}} N \in \mathbf{K}_\lambda$, no subset of $\mathbf{S}_*(N)$ of size $\leq \lambda$ is $\mathbf{S}_*$-inevitable,*

*then $\mathbb{I}(\mathbf{K}, \lambda^+) = 2^{\lambda^+}$.*

We show how to get stability for $\lambda$-algebraic types.

**Theorem 69.** *($2^\lambda < 2^{\lambda^+}$) If $\mathbb{I}(\mathbf{K}, \lambda) = 1 \leq \mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$, then $\mathbf{K}$ is stable for $\lambda$-algebraic types in $\lambda$.*

*Proof.* Assume for the sake of contradiction that there is $M \in \mathbf{K}_\lambda$ such that $|\mathbf{S}^{\lambda-al}(M)| \geq \lambda^+$. Observe that $\mathbf{K}$ has amalgamation and no maximal models in $\lambda$ by Fact 49.

We show that conditions (1) to (3) of Fact 68 hold for $\mathbf{S}_* = \mathbf{S}^{\lambda-al}$. This is enough as Fact 68 implies that $\mathbb{I}(\mathbf{K}, \lambda^+) = 2^{\lambda^+}$ and we assumed that $\mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$. Condition (1) is Proposition 42, Condition (2) is Lemma 67 and Condition (3).(a) is our assumption that $|\mathbf{S}^{\lambda-al}(M)| \geq \lambda^+$. So we only need to show Condition (3).(b). Let $M \leq_{\mathbf{K}} N \in \mathbf{K}_\lambda$.

Claim: There is no $\Gamma \subseteq \mathbf{S}^{\lambda-al}(N)$ such that $|\Gamma| \leq \lambda$ and $\Gamma$ is $\mathbf{S}^{\lambda-al}$-inevitable.

Proof of Claim: Otherwise, suppose there exists such $\Gamma$. If we show that Condition (2) of Fact 44 for $\mathbf{S}_* = \mathbf{S}^{\lambda-al}$ holds, we would be done as Fact 44 would imply that $|\mathbf{S}^{\lambda-al}(M)| \leq \lambda$ which contradicts the assumption that $|\mathbf{S}^{\lambda-al}(M)| \geq \lambda^+$. Let $L \in \mathbf{K}_\lambda$. Then there is $f : L \cong N$ an isomorphism by $\lambda$-categoricity. Using $f$ we can copy $\Gamma$ to a $\Gamma_L \subseteq \mathbf{S}^{na}(L)$ such that $|\Gamma_L| \leq \lambda$ and $\Gamma_L$ is $\mathbf{S}^{\lambda-al}$-inevitable as $\Gamma$ is $\mathbf{S}^{\lambda-al}$-inevitable. $\dagger_{\text{Claim}}$ $\square$

We obtain the following positive answer to Grossberg's question for small cardinals assuming a mild locality condition for Galois types and without any stability assumption.

**Theorem 70.** *($2^\lambda < 2^{\lambda^+}$) Suppose $\lambda^+ < 2^{\aleph_0}$ and Fact 44 holds. Assume $\mathbb{I}(\mathbf{K}, \lambda) = 1 \leq \mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$. If $\mathbf{K}$ is $(< \lambda^+, \lambda)$-local, then $\mathbf{K}$ has a model of cardinality $\lambda^{++}$.*

*Proof.* It follows that $\mathbf{K}$ has amalgamation in $\lambda$ by Fact 49 and it is clear that $\mathbf{K}$ has no maximal model in $\lambda$. Moreover, $\mathbf{K}$ is stable for $\lambda$-algebraic types in $\lambda$ by Theorem 69. Therefore, $\mathbf{K}$ has a model of cardinality $\lambda^{++}$ by Lemma 64. □

**Remark 71.** *For AECs $\mathbf{K}$ with $\mathrm{LS}(\mathbf{K}) = \aleph_0$ and $\lambda = \aleph_0$. The previous result is known for universal classes (even without the assumption that $2^{\aleph_0} > \aleph_1$) [MAV18, 3.3], but it is new for $(< \aleph_0, \aleph_0)$-tame AECs. For $\lambda > \aleph_1$, the result is new even for universal classes.*

**Remark 72.** *The set theoretic assumption that $\lambda^+ < 2^{\aleph_0}$ can be replaced by the model theoretic assumption that $|\mathbf{S}^{na}(M)| < 2^{\aleph_0}$ for every $M \in \mathbf{K}_\lambda$ by using Theorem 61 instead of Lemma 64.*

Let us consider the following property on chains of types:

**Definition 73.** *A type family $\mathbf{S}_*$ is $\lambda$-compact if for every limit ordinal $\delta < \lambda^+$, for every $\langle M_i \in \mathbf{K}_\lambda : i < \delta \rangle$ an increasing continuous chain and for every coherent sequence of types $\langle p_i \in \mathbf{S}_*(M_i) : i < \delta \rangle$, there is an upper bound $p \in \mathbf{S}_*(\bigcup_{i<\delta} M_i)$ to the sequence such that $\langle p_i \in \mathbf{S}_*(M_i) : i < \delta + 1 \rangle$ is a coherent sequence.*

**Remark 74.** *For every $M \in \mathbf{K}_\lambda$, let*

$$\mathbf{S}^{\lambda-sunq}(M) = \{p \in \mathbf{S}^{\lambda-unq}(M) : p \text{ has the } \lambda\text{-strong extension property}\}.$$

*The limit step of Theorem 61 basically shows that if $\mathbf{K}$ is $(< \lambda^+, \lambda)$-local then $\mathbf{S}^{\lambda-sunq}$ is $\lambda$-compact.*

The locality assumption on types and cardinal arithmetic assumption that $\lambda^+ < 2^{\aleph_0}$ can be dropped from Theorem 70 if instead we assume that the larger class of types $\mathbf{S}^{\lambda-ext}$ is $\lambda$-compact. The result still uses Fact 44.

**Corollary 75.** *($2^\lambda < 2^{\lambda^+}$) Assume $1 = \mathbb{I}(\mathbf{K}, \lambda) \leq \mathbb{I}(\mathbf{K}, \lambda^+) < 2^\lambda$, and. If $\mathbf{S}^{\lambda-ext}$ is $\lambda$-compact, then $\mathbf{K}$ has a model of cardinality $\lambda^{++}$.*

*Proof.* The proof is similar to that of Theorem 61, except that in the construction we only require that $p_i$ has the $\lambda$-extension property instead of being a $\lambda$-unique type. At limit stage we can do the construction using that the types only have the $\lambda$-extension property because of the assumption that $\mathbf{S}^{\lambda-ext}$ is $\lambda$-compact. □

# Chapter 4

# On stability and existence of models in local AECs

## 4.1 Introduction

In this chapter we presents results in a forthcoming paper with Marcos Mazari-Armida and Sebastien Vasey. Throughout the chapter $(\mathbf{K}, \leq_{\mathbf{K}})$ is an abstract elementary class and $\lambda$ is a cardinal such that $\lambda \geq LS(\mathbf{K})$.

When $\mathbf{K}$ is elementary, Morley, in an intermediate step of proving his categoricity theorem [Mor65], showed that categoricity in an uncountable cardinal implies stability in $\aleph_0$. For abstract elementary classes it is not known. In the third section of this chapter we provide a partial answer to this problem.

First we use [She09a, 2.8] to prove that the abstract elementary class $\mathbf{K}$ is almost stable in $\lambda$. Then, using almost stability, we show that there is a minimal type, and hence stability in $\lambda$:

**Theorem 76.** *Suppose that $\lambda^+ < 2^\lambda < 2^{\lambda^+}$. Assume $\mathbf{K}$ is categorical in $\lambda$ and $\lambda^+$, $\mathbf{K}_{\lambda^{++}} \neq \emptyset$ and $|\mathbf{S}^{\neg\lambda^+-min}(M)| \leq \lambda^{+}$*[1] *for the unique model $M \in \mathbf{K}_\lambda$. Then $\mathbf{K}$ is stable in $\lambda$.*

We also obtain a similar result under slightly different assumptions using machinery from [She09a]:

[1]See Definition 45

**Theorem 77.** *Suppose that* $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$. *Assume* $\mathbf{K}$ *is categorical in* $\lambda$ *and* $\lambda^+$, $\mathbf{K}_{\lambda^{++}} \neq \emptyset$, *and* $|\mathbf{S}(N)| < 2^{\lambda^+}$ *for the unique model* $N \in \mathbf{K}_{\lambda^+}$. *Then* $\mathbf{K}$ *is stable in* $\lambda$.

There are not many results of this kind. One is [BLS23], where the authors prove stability below categoricity and existence for atomic classes which is the following theorem:

**Theorem 78.** *If an atomic class is categorical in* $\aleph_1$ *and has a model in* $(2^{\aleph_0})^+$, *then the class is stable in* $\aleph_0$.

These results are interesting not only because they generalize the result of Morley, but also for their usage in proving other theorems, where stability is usually a key step. Comparing our result to [BLS23], note that our result does not need $\lambda$ to be $\aleph_0$ while working in the more general context of abstract elementary classes despite the extra assumptions on the number of types. Other results such as [Vas16a] and [BGVV17] assume that $\mathbf{K}$ has arbitrarily large models, which we do not.

The fourth section is dedicated to prove existence from stability. Noting that continuity of splitting follows from the locality assumption, we prove that $\mathbf{K}$ has a model in $\lambda^{++}$ assuming amalgamation and stability in $\lambda$.

**Theorem 79.** *Let* $\mathbf{K}$ *be an AEC and let* $\lambda \geq \mathrm{LS}(\mathbf{K})$. *If* $\mathbf{K}$ *has amalgamation in* $\lambda$, *no maximal model in* $\lambda$ *and is stable in* $\lambda$, *and splitting is continuous in* $\lambda$, *then* $\mathbf{K}$ *has a model in* $\lambda^{++}$.

This also provides a partial answer to Grossberg's successive categoricity question:

**Theorem 80.** *Suppose that* $2^\lambda < 2^{\lambda^+}$ *and that* $\lambda^+ < 2^{\aleph_0}$. *If* $\mathbb{I}(\mathbf{K}, \lambda) = 1 \leq \mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$, *and* $\mathbf{K}$ *is* $(< \lambda^+, \lambda)$*-local, then* $\mathbf{K}$ *has a model of cardinality* $\lambda^{++}$.

Note that we can get a model in $\lambda^{++}$ without assuming $\lambda^+ < 2^{\aleph_0}$, despite assuming stability in $\lambda$.

Combining the previous results, we obtain an equivalence between stability and existence:

**Theorem 81.** *Suppose* $\lambda^+ < 2^\lambda < 2^{\lambda^+}$. *Assume* $\mathbf{K}$ *is categorical in* $\lambda$ *and* $\lambda^+$, $\mathbf{K}$ *is* $(< \lambda^+, \lambda)$*-local and* $\mathbf{K}$ *is almost stable for non-*$\lambda^+$*-minimal types in* $\lambda$. *The following are equivalent.*

1. $\mathbf{K}$ *has a model in* $\lambda^{++}$.
2. $\mathbf{K}$ *is stable in* $\lambda$.

Finally, we prove a categoricity result. With tamness and weak amalgamation (see

Definition 106), we build a w-good $\lambda^+$-frame (see [MA20]) with density, and then extend it to a w-good $[\lambda^+, \infty)$-frame to show that $\mathbf{K}$ is categorical in every $\mu \in [\lambda, \infty)$:

**Theorem 82.** *Let $\mathbf{K}$ be an AEC with weak amalgamation and let $\lambda \geq \mathrm{LS}(\mathbf{K})$ be such that $\mathbf{K}$ is $\lambda$-tame. Assume $\mathbf{K}$ has amalgamation in $\lambda$, $\mathbf{K}$ is stable in $\lambda$, and splitting is continuous in $\lambda$. If $\mathbf{K}$ is categorical in $\lambda$ and $\lambda^+$, then $\mathbf{K}$ is categorical in all $\mu \geq \lambda$.*

This can be seen as a generalization of the result of Grossberg and VanDieren [GV05], which assumes arbitrarily large models and amalgamation. Indeed, we can replace locality and the set-theoretic assumptions by model-theoretic ones that are easily derivable in the Grossberg-VanDieren context: it suffices to assume stability in $\lambda$, amalgamation in $\lambda$, and continuity of splitting in $\lambda$. In [Vas22], it was already observed that the Grossberg-VanDieren result carries through if only weak amalgamation is assumed, but here we do not even assume arbitrarily large models. Moreover, this is the second time that an application of w-good frames has been discovered.

## 4.2 Getting stability in $\lambda$

### 4.2.1 Almost stable in $\lambda$

The following theorem is [She09a, §VI.2.11] and [JS13, 2.5.8]. We include the details for the sake of completeness.

**Fact 83.** *Suppose $\mathbf{K}$ has amalgamation and no maximal model in $\lambda$. Let $\mathbf{S}_*$ be a $\leq_{\mathbf{K}}$-type kind and hereditary. Suppose that for all $M \in \mathbf{K}_\lambda$ there is $\Gamma_M \subseteq \mathbf{S}_*(M)$ such that $|\Gamma_M| \leq \lambda^+$ and $\Gamma_M$ is $\mathbf{S}_*$-inevitable. Then there is a model saturated for $\mathbf{S}_*$-types in $\lambda^+$ above $\lambda$. In particular, for all $M \in \mathbf{K}_\lambda$, $|\mathbf{S}_*(M)| \leq \lambda^+$.*

*Proof.* Fix a bijection $g : \lambda^+ \times \lambda^+ \to \lambda^+$. We build $\langle M_i : i < \lambda^+ \rangle$ and $\langle p_{i,j} : i, j < \lambda^+ \rangle$ such that:

1. $M_i \in \mathbf{K}_\lambda$ for all $i < \lambda^+$;
2. $\langle M_i : i < \lambda^+ \rangle$ is increasing and continuous;
3. $\{p_{i,j} : j < \lambda^+\} = \Gamma_{M_i}$ for all $i < \lambda^+$;
4. $M_{i+1}$ realizes $p_{g(\epsilon)}$, where $\epsilon$ is the least such that $g(\epsilon) = (\alpha, \beta)$, $\alpha \leq i$, and $p_{\alpha,\beta}$ is not realized in $M_i$.

We now claim that $M_{\lambda^+} := \bigcup_{i<\lambda^+} M_i$ is saturated for $\mathbf{S}_*$-types above $\lambda$. It suffices to show that for any $M_0 <_{\mathbf{K}} N \in \mathbf{K}_\lambda$, $a \in |N| - |M_0|$, $p = \mathbf{gtp}(a/M_0, N) \in \mathbf{S}_*(M_0)$, $p$ is realized in $M_{\lambda^+}$. We build: $\langle N_i : i < \lambda^+ \rangle$, $\langle \alpha_i : i < \lambda^+ \rangle$ and $f_i : M_{\alpha_i} \to N_i$ such that:

1. $N_i \in \mathbf{K}_\lambda$ for all $i < \lambda^+$;
2. $\alpha_i < \lambda^+$ for all $i < \lambda^+$;
3. $\langle N_i : i < \lambda^+ \rangle$ is increasing and continuous;
4. $\langle f_i : i < \lambda^+ \rangle$ is increasing and continuous;
5. $\langle \alpha_i : i < \lambda^+ \rangle$ is increasing and continuous;
6. $N_0 = N$;
7. $\alpha_0 = 0$;
8. $f_0 = id_{M_0}$;
9. $\| N_i | - | f_i[M_{\alpha_i}] \| \geq 1$;
10. For each $i < \lambda^+$ there is $b \in |M_{\alpha_{i+1}}| - |M_{\alpha_i}|$ such that $f_{i+1}(b) \in |N_i|$.

We carry out the construction by induction on $i < \lambda^+$. The base is clear. At successor $i+1$, if $a \in f_i[M_{\alpha_i}]$, then already $M_{\alpha_i}$ realizes $\mathbf{gtp}(a/M_0, N)$, and we are done. We will prove that this must happen for some $i$.

Otherwise we continue the construction. Since $\| N_i | - | f[M_{\alpha_i}] \| \geq 1$ and $\mathbf{gtp}(a/f_i[M_{\alpha_i}], N_i) \in \mathbf{S}_*(f_i[M_{\alpha_i}])$ because $S^*$ is hereditary, by inevitability there is $b \in |N_i| - |f_i[M_{\alpha_i}]|$ such that $\mathbf{gtp}(b/f_i[M_{\alpha_i}], N_i) = f_i(p)$ for some $p \in \Gamma_{M_{\alpha_i}}$. (Why? note that the image of an $\mathbf{S}_*$-inevitable set remains $\mathbf{S}_*$-inevitable, so $\{f_i(q) : q \in \Gamma_{M_{\alpha_i}}\}$ is $\mathbf{S}_*$-inevitable.) There is $\alpha_{i+1}$ such that $M_{\alpha_{i+1}}$ realizes $p$ by condition (4) of the construction of $\langle M_i : i < \lambda^+ \rangle$, so we can find $f_{i+1}$ and $N_{i+1}$ such that

$$\begin{array}{ccc} N_i & \longrightarrow & N_{i+1} \\ {\scriptstyle f_i}\uparrow & & {\scriptstyle f_{i+1}}\uparrow \\ M_{\alpha_i} & \longrightarrow & M_{\alpha_{i+1}} \end{array}$$

commutes with $b = f_{i+1}(c)$ for some $c \in |M_{\alpha_{i+1}}|$ and $\| N_{i+1} - | f_{i+1}[M_{\alpha_{i+1}}] \| \geq 1$

When $i$ is a limit, choose $N_i := \bigcup_{j<i} N_j$ and $f_i := \bigcup_{j<i} f_j$. Observe $\| N_i | - | f[M_{\alpha_i}] \| \geq 1$ as otherwise $a \in f_j[M_{\alpha_j}]$ for $j < i$ and we would have stopped the construction.

Finally let $N_{\lambda^+} := \bigcup_{i<\lambda^+} N_i$, $f := \bigcup_{i<\lambda^+} f_i$, and $N^* := f[M_{\lambda^+}]$. Now $\langle f_i[M_{\alpha_i}] : i < \lambda^+ \rangle$ and $\langle N_i \cap N^* : i < \lambda^+ \rangle$ are two resolutions of $N^*$, so $f_i[M_{\alpha_i}] = N_i \cap N^*$ for some $i$. Then

$f_i[M_{\alpha_i}] \subseteq N_i \cap f_{i+1}[M_{\alpha_{i+1}}] \subseteq N_i \cap N^* = f_i[M_{\alpha_i}]$, but this contradicts condition (10) of $\langle N_i : i < \lambda^+ \rangle$.

Thus, the construction of $\langle N_i : i < \lambda^+ \rangle$ and $\langle f_i : i < \lambda^+ \rangle$ is not possible, so it must be that for some $i < \lambda^+$, $a \in f_i[M_{\alpha_i}]$. This shows that $M_{\lambda^+}$ realizes all of $\mathbf{S}_*(M_0)$. In fact $M_{\lambda^+}$ realizes $\mathbf{S}_*(M_i)$ for all $i < \lambda^+$. Since any $\leq_{\mathbf{K}}$-substructure of $M_{\lambda^+}$ is contained in some $M_i$, we conclude that it is saturated above $\lambda$ in $\lambda^+$ for $\mathbf{S}_*$-types. $\square$

**Theorem 84.** *Assume $2^\lambda < 2^{\lambda^+}$. Suppose $\mathbf{K}$ has amalgamation and no maximal model in $\lambda$, categoricity in $\lambda$ and $\lambda^+$, and $|\mathbf{S}^{\neg\lambda^+-min}(M)| \leq \lambda^+$ for the unique model $M \in \mathbf{K}_\lambda$. Then $\mathbf{K}$ is almost stable in $\lambda$.*

*Proof.* Assume for the sake of contradiction that $|\mathbf{S}(M)| > \lambda^+$. Then $|\mathbf{S}^{\lambda^+-min}(M)| > \lambda^+$ as $|\mathbf{S}^{\neg\lambda^+-min}(M)| \leq \lambda^+$. Since $|\mathbf{S}^{\lambda^+-min}(M)| \geq \lambda^{++}$, no subset of $\mathbf{S}^{\lambda^+-min}(M)$ of size $\leq \lambda^+$ is $\mathbf{S}^{\lambda^+-min}$-inevitable by categoricity in $\lambda$ and Fact 83. We build $\langle M_\eta : \eta \in {}^{<\lambda^+}2 \rangle$ and $\langle \Gamma_\eta : \eta \in {}^{<\lambda^+}2 \rangle$ such that:

1. $M_\eta \in \mathbf{K}_\lambda$ for all $\eta \in {}^{<\lambda^+}2$;
2. $\Gamma_\eta \subseteq \bigcup_{j \leq i} \mathbf{S}^{\lambda^+-min}(M_{\eta \restriction i})$ for all $\eta \in {}^i 2$, $i < \lambda^+$;
3. $|\Gamma_\eta| \leq \lambda^+$ for all $\eta \in {}^{<\lambda^+}2$;
4. $M_\eta$ omits all types in $\Gamma_\eta$ for all $\eta \in {}^{<\lambda^+}2$.
5. If $\eta < \nu \in {}^{<\lambda^+}2$, then $M_\eta \leq_{\mathbf{K}} M_\nu$ and $\Gamma_\eta \subseteq \Gamma_\nu$;
6. $\lambda + i \leq |M_\eta| \leq \lambda + \lambda \cdot i$ for $\eta \in {}^i 2$, $i < \lambda^+$;
7. For all $\eta$, $M_{\eta^\frown \ell}$ realizes a type over $M_\eta$ from $\Gamma_{\eta^\frown(1-\ell)}$ for $\ell = 0, 1$;
8. For all $\eta$, $\{\mathbf{gtp}(a/M_\eta, M_{\eta^\frown 1-\ell}) \in \mathbf{S}^{\lambda^+-min}(M_\eta) : a \in M_{\eta^\frown(1-\ell)}\} \subseteq \Gamma_{\eta^\frown \ell}$ for $\ell = 0, 1$.

Construction: We build everything by induction on the $i$, the length of $\eta$. Let $M_{\langle\rangle}$ be the unique model in $\mathbf{K}_\lambda$ and $\Gamma_{\langle\rangle} := \emptyset$. At limits let $M_\eta := \bigcup_{j<i} M_{\eta \restriction j}$ and $\Gamma_\eta := \bigcup_{j<i} \Gamma_{\eta \restriction j}$. At successor $i+1$, let $M_{\eta^\frown 0}$ be any $\leq_{\mathbf{K}}$-extension of $M_\eta$ such that:

1. $M_{\eta^\frown 0}$ omits
$$\Gamma'_\eta := \bigcup_{j<i} \{q \in \mathbf{S}^{na}(M_\eta) : q \restriction M_{\eta \restriction j} \in \Gamma_{\eta \restriction j}\}.$$
2. some $a \in |M_{\eta^\frown 0}|$ satisfies that $\mathbf{gtp}(a/M_\eta, M_{\eta^\frown 0}) \in \mathbf{S}^{\lambda^+-min}(M_\eta)$.

Note that $M_{\eta^\frown 0}$ exists as we can omit any set of $\lambda^+$-minimal types of size $\leq \lambda^+$ while realizing at least one type that is $\lambda^+$-minimal as no set of $\lambda^+$ types is $\mathbf{S}^{\lambda^+-min}$-inevitable. This is possible as each type in $\Gamma_{\eta\restriction j}$ has $\leq \lambda^+$ extensions to $\mathbf{S}(M_\eta)$ so $|\Gamma'_\eta| \leq \lambda^+$ and every type in $\Gamma'_\eta$ is $\lambda^+$-minimal. Arrange that $|M_{\eta^\frown 0}|$ is an ordinal $|M_\eta|+\kappa$ for some $1 \leq \kappa \leq \lambda$. By induction $\lambda+i \leq |M_\eta| \leq \lambda+\lambda\cdot i$, so we obtain $\lambda+(i+1) \leq |M_{\eta^\frown 0}| \leq \lambda+\lambda\cdot i+\lambda = \lambda+\lambda\cdot(i+1)$.

Let

$$\Gamma'_{\eta^\frown 1} := \Gamma'_\eta \cup \{\mathbf{gtp}(a/M_\eta, M_{\eta^\frown 0}) : \mathbf{gtp}(a/M_\eta, M_{\eta^\frown 0}) \in \mathbf{S}^{\lambda^+-min}(M_\eta)\}$$

and $M_{\eta^\frown 1}$ be any $\mathbf{K}$-extension of $M_\eta$ such that:

1. $\lambda + i \leq |M_{\eta^\frown 1}| \leq \lambda + \lambda \cdot (i+1)$;
2. $M_{\eta^\frown 1}$ omits $\Gamma'_{\eta^\frown 1}$;
3. some $a \in |M_{\eta^\frown 1}|$ satisfies that $\mathbf{gtp}(a/M_\eta, M_{\eta^\frown 1}) \in \mathbf{S}^{\lambda^+-min}(M_\eta)$.

$M_{\eta^\frown 1}$ exists because $\Gamma'_{\eta^\frown 1}$ is not $\mathbf{S}^{\lambda^+-min}$-inevitable. One can check that the other requirements are satisfied as for $M_{\eta^\frown 0}$.

Finally, let

$$\Gamma_{\eta^\frown 0} := \Gamma_\eta \cup \{\mathbf{gtp}(a/M_\eta, M_{\eta^\frown 1}) : \mathbf{gtp}(a/M_\eta, M_{\eta^\frown 1}) \in \mathbf{S}^{\lambda^+-min}(M_\eta)\},$$

and

$$\Gamma_{\eta^\frown 1} := \Gamma_\eta \cup \{\mathbf{gtp}(a/M_\eta, M_{\eta^\frown 0}) : \mathbf{gtp}(a/M_\eta, M_{\eta^\frown 0}) \in \mathbf{S}^{\lambda^+-min}(M_\eta)\}.$$

We only check requirement (4) of the construction as the others are easy to check. We show that $M_{\eta^\frown 0}$ omits $\Gamma_{\eta^\frown 0}$. Let $p \in \Gamma_{\eta^\frown 0}$. Assume for the sake of contradiction that $p$ is realized by $a \in M_{\eta^\frown 0}$. $p$ cannot be of the form $\mathbf{gtp}(b/M_\eta, M_{\eta^\frown 1})$ since it lies in $\Gamma_{\eta^\frown 0}$ by requirement (8) and hence must be omitted by $M_{\eta^\frown 0}$. So $p \in \Gamma_\eta$, then $M_{\eta^\frown 0}$ omitted all non-algebraic extensions of $p$ as they are in $\Gamma'_\eta$, and any algebraic extension of $p$ cannot be realized since it must lie in $M_\eta$, but $M_\eta$ omits $\Gamma_\eta$ by induction hypothesis. Similarly $M_{\eta^\frown 1}$ omits $\Gamma_{\eta^\frown 1}$.

Enough Let $C := \{\delta < \lambda^+ : \lambda + \lambda \cdot \delta = \delta = \lambda + \delta\}$. Note that $C$ is a club. By Fact 48 there are disjoint stationary sets $S_\gamma \subseteq \lambda^+$ such that $\Phi_{\lambda^+}(S_\gamma)$ holds for all $\gamma < \lambda^+$.

We denote the zero sequence in ${}^{\lambda^+}2$ by $\bar{0}$. For $\delta \in C$, and $\eta \in {}^{\delta}2$ and $h : \delta \to \delta$, define

$$F(\eta,h) := \begin{cases} 1 & \text{if } h: M_\eta \to M_{\bar{0}\restriction\delta} \text{ and for some } \beta<\lambda^+ \text{ and } g: M_{\eta^\frown 0} \to M_{\bar{0}\restriction\beta} \text{ extending } h \\ & \text{there are } a \in |M_{\eta^\frown 0}| - |M_\eta|, b \in |M_{\bar{0}\restriction\beta}| - |M_{\bar{0}\restriction\delta}| \text{ such that} \\ & \begin{array}{ccc} M_{\eta^\frown 0} & \xrightarrow{g} & M_{\bar{0}\restriction\beta} \\ \uparrow & & \uparrow \\ M_\eta & \xrightarrow{h} & M_{\bar{0}\restriction\delta} \end{array} \text{ commutes with } g(a)=b \\ & \text{and } \mathbf{gtp}(a/M_\eta, M_{\eta^\frown 0}) \in \mathbf{S}^{\lambda^+ - min}(M_\eta) \\ 0 & \text{otherwise.} \end{cases}$$

For all $\gamma < \lambda^+$, by $\Phi_{\lambda^+}(S_\gamma)$ there is $g_\gamma : \lambda^+ \to 2$ such that for all $\eta \in {}^{\lambda^+}2$ and $h : \lambda^+ \to \lambda^+$.

$$\{\delta \in S\gamma : F(\eta \restriction \delta, h \restriction \delta) = g_\gamma(\delta)\}$$

is stationary.

For each $X \subseteq \lambda^+$ define

$$\eta_X(\delta) := \begin{cases} g_\gamma(\delta) & \text{if } \delta \in S_\gamma, \gamma \in X, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathbb{I}(\mathbf{K}, \lambda^+) = 1$, the following claim would give us a contradiction.

Claim: For $X \neq \emptyset \subseteq \lambda^+$, there is no $h : M_{\eta_X} \cong M_{\bar{0}}$.

Proof of Claim. Assume there are such $X$ and $h$. Let $\gamma \in X$. Then

$$D := \{\delta < \lambda^+ : h \restriction \delta : \delta \to \delta\}$$

is a club. Let $S'_\gamma = \{\delta \in S\gamma : F(\eta_X \restriction \delta, h \restriction \delta) = g_\gamma(\delta)\}$ be the stationary set obtained from $\eta_X, h$.

Let $\delta \in S'_\gamma \cap D \cap C$. Observe that for all $\eta \in {}^\delta 2$, $\delta = \lambda + \delta \leq |M_\eta| \leq \lambda + \lambda \cdot \delta = \delta$, i.e. $|M_\eta| = \delta$. Since $h \restriction \delta : \delta \to \delta$, $h$ is a $\mathbf{K}$-embedding from $M_{\eta_X \restriction \delta}$ to $M_{\bar{0}\restriction\delta}$.

We divide the proof into two cases:

**Case 1:** $\eta_X(\delta) = 1$. Then $g_\gamma(\delta) = F(\eta_X \restriction \delta, h \restriction \delta) = 1$, so the following diagram commutes

$$\begin{CD} M_{(\eta_X\restriction\delta)^\frown 0} @>g>> M_{\bar{0}\restriction\beta} \\ @AAA @AAA \\ M_{\eta_X\restriction\delta} @>h>> M_{\bar{0}\restriction\delta} \end{CD} \tag{4.1}$$

with

$$g(a) = b \tag{$\dagger_1$}$$

for some $\delta < \beta < \lambda^+$, $a \in |M_{(\eta_X\restriction\delta)^\frown 0}| - |M_{\eta_X\restriction\delta}|$ and $b \in |M_{\bar{0}\restriction\beta}| - |M_{\bar{0}\restriction\delta}|$ such that

$$\mathbf{gtp}(a/M_{\eta\restriction\delta}, M_{(\eta_X\restriction\delta)^\frown 0}) \in \mathbf{S}^{\lambda^+ -min}(M_{\eta_X\restriction\delta}).$$

Since $h : M_{\eta_X} \cong M_{\bar{0}}$ and $b \in |M_{\bar{0}\restriction\beta}|$, there are $\delta < \beta' < \lambda^+$ and $c \in |M_{\eta_X\restriction\beta'}|$ such that the following diagram commutes:

$$\begin{CD} M_{\eta_X\restriction\delta} @>h>> M_{\bar{0}\restriction\delta} \\ @VV{id}V @VV{id}V \\ M_{\eta_X\restriction\beta'} @>h>> M_{\bar{0}\restriction\beta'} \end{CD} \tag{4.2}$$

with

$$h(c) = b \tag{$\dagger_2$}$$

Note that $c \notin |M_{\eta_X\restriction\delta}|$ since $h(c) = b \notin |M_{\bar{0}\restriction\delta}|$. Without loss of generality assume $\beta = \beta'$ as we can take them arbitrarily large as long as $M_{\bar{0}\restriction\beta}$ contains $b$ and $h[M_{\eta_X\restriction\beta'}]$.

Now we put the two diagrams together:

$$\begin{array}{ccccc} M_{(\eta_X\restriction\delta)^\frown 0} & & & \overset{g}{\searrow} & \\ \uparrow & & & & \\ M_{\eta_X\restriction\delta} & \xrightarrow{h} & M_{\bar{0}\restriction\delta} & \longrightarrow & M_{\bar{0}\restriction\beta} \\ \downarrow & & & \underset{h}{\nearrow} & \\ M_{\eta_X\restriction\beta} & & & & \end{array} \tag{4.3}$$

Now the outer diagram

$$\begin{CD} M_{(\eta_X\restriction\delta)^\frown 0} @>g>> M_{\bar{0}\restriction\beta} \\ @A{id}AA @A{h}AA \\ M_{\eta_X\restriction\delta} @>id>> M_{\eta_X\restriction\beta} \end{CD} \tag{4.4}$$

commutes with $g(a) = b = h(c)$ by $\dagger_1$ and $\dagger_2$, so

$$\mathbf{gtp}(a/M_{\eta_X\restriction\delta}, M_{(\eta_X\restriction\delta)^\frown 0}) = \mathbf{gtp}(c/M_{\eta_X\restriction\delta}, M_{\eta_X\restriction\beta}).$$

This is impossible, since $\mathbf{gtp}(a/M_{\eta_X\restriction\delta}, M_{(\eta_X\restriction\delta)^\frown 0}) \in \Gamma_{(\eta_X\restriction\delta)^\frown 1}$ by requirement (8), and $\Gamma_{(\eta_X\restriction\delta)^\frown 1} \subseteq \Gamma_{\eta_X\restriction\beta}$ is omitted by $M_{\eta_X\restriction\beta}$ by requirements (4) and (5). This finishes **Case 1**.

**Case 2:** $\eta_X(\delta) = g_\gamma(\delta) = F(\eta_X \restriction \delta, h \restriction \delta) = 0$. We now show that

$$F(\eta_X \restriction \delta, h \restriction \delta) = 1$$

so that this case is not possible.

Let $a \in |M_{(\eta_X\restriction\delta)^\frown 0}| - |M_{\eta_X\restriction\delta}|$ such that $\mathbf{gtp}(a/M_{\eta_X\restriction\delta}, M_{(\eta_X\restriction\delta)^\frown 0}) \in \mathbf{S}^{\lambda^+-min}(M_{\eta_X\restriction\delta})$. We can find such $a$ by the condition (7) of the construction. Since $b := h(a) \in |M_{\bar{0}}|$, find $\delta < \beta < \lambda^+$ such that

$$\begin{array}{ccc} M_{(\eta_X\restriction\delta)^\frown 0} & \xrightarrow{h} & M_{\bar{0}\restriction\beta} \\ \uparrow & & \uparrow \\ M_{\eta_X\restriction\delta} & \xrightarrow{h} & M_{\bar{0}\restriction\delta} \end{array}$$

commutes with $h(a) = b$ and $b \in |M_{\bar{0}\restriction\beta}| - |M_{\bar{0}\restriction\delta}|$ (since $a \notin |M_{\eta_X\restriction\delta}|$). Thus $F(\eta_X \restriction \delta, h \restriction \delta) = 1$, contradiction.

□

### 4.2.2 Stable in $\lambda$

We turn our attention to obtain stability in $\lambda$.

**Lemma 85** ([She96, 6.3])**.** *If $\lambda^+ < 2^\lambda$, then there is a tree with $\leq \lambda$ nodes and $\kappa \leq \lambda$ levels with at least $\lambda^{++}$ branches of length $\kappa$.*

*Proof.* Let $\kappa$ be the least cardinal such that $2^\kappa > \lambda^+$. Consider the tree ${}^{<\kappa}2$. If $2^{<\kappa} \leq \lambda$ this tree is enough. Indeed, its set of branches of length $\kappa$ is just ${}^{\kappa}2$, which is of cardinality $2^\kappa > \lambda^+$.

Now suppose $2^{<\kappa} > \lambda$. Since $2^\lambda > \lambda^+$, $\kappa \leq \lambda$. Then $2^{<\kappa} = \lambda^+$ by the assumptions that $2^{<\kappa} > \lambda$ and that $\kappa$ is minimal. Write $2^{<\kappa} = \bigcup_{i<\lambda^+} B_i$, $B_i$ increasing with $i$, $|B_i| \leq \lambda$. For each $\eta \in {}^{\kappa}2$ and each $\alpha < \kappa$, $\eta \restriction \alpha \in B_i$ for some $i$. Then there is $j(\eta)$ such that

$\eta \restriction \alpha \in B_{j(\eta)}$ happens for cofinally many $\alpha < \kappa$. As for each $\alpha < \kappa$ there is $k_\alpha$ such that $\eta \restriction \alpha \in B_{k_\alpha}$. $\sup\{k_\alpha : \alpha < \kappa\} < \lambda^+$, so take $j(\eta)$ to be this supremum. Consider the map $\eta \mapsto j(\eta)$ from ${}^\kappa 2$ to $\lambda^+$. By the pigeonhole principle there is $j^*$ such that $|\{\eta \in {}^{<\kappa}2 : j(\eta) < j^*\}| \geq \lambda^{++}$. Note that $\{\eta \in {}^\kappa 2 : j(\eta) < j^*\}$ is the set of branches of length $\kappa$ of the tree $T := \{\eta \restriction \alpha : \alpha < \ell(\eta), \eta \in B_{j^*}\}$. Moreover, $|T| \leq \kappa \cdot |B_{j^*}| \leq \lambda \cdot \lambda = \lambda$. □

**Fact 86** ([She09a, VI.2.5.(1),(3)]). *If $M \in \mathbf{K}_\lambda$, and there is no minimal type above $p \in \mathbf{S}^{na}(M)$, then there is $N \in \mathbf{K}_\lambda$ such that $p$ has $\lambda^+$ extensions to $\mathbf{S}^{na}(N)$ and $p$ has the extension property. Note that above these extensions there are not minimal types either.*

**Fact 87.** *([She09a, VI.5.3(1)]) Suppose $2^\lambda < 2^{\lambda^+}$. Assume that $\mathbf{K}$ is categorical in $\lambda$ and $\lambda^+$ and that $\mathbf{K}_{\lambda^{++}} \neq \emptyset$. If there is an minimal type over (the unique) $M \in \mathbf{K}_\lambda$, then there is an inevitable one.*

**Fact 88** ([She09a, VI.5.8(1)]). *Assume $\mathbf{K}$ is categorical in $\lambda$, has amalgamation in $\lambda$, and has a model in $\lambda^{++}$. If there is an inevitable type over $M \in \mathbf{K}_\lambda$, then $\mathbf{K}$ is stable in $\lambda$.*

We obtain the main result of this section which is the forward direction of the main theorem mentioned in the introduction:

**Theorem 89.** *Suppose $\lambda^+ < 2^\lambda < 2^{\lambda^+}$. Assume $\mathbf{K}$ is categorical in $\lambda$ and $\lambda^+$, $\mathbf{K}_{\lambda^{++}} \neq \emptyset$ and $|\mathbf{S}^{\neg\lambda^+-min}(M)| \leq \lambda^+$ for the unique model $M \in \mathbf{K}_\lambda$. Then $\mathbf{K}$ is stable in $\lambda$.*

*Proof.* We show that there is a minimal type in $\mathbf{K}_\lambda$. This is enough as it implies the existence of an inevitable minimal type by Fact 87, which in turn implies that $\mathbf{K}$ is stable in $\lambda$ by Fact 88.

Assume for the sake of contradiction that there is not a minimal type in $\mathbf{K}_\lambda$. Build $\langle M_i : i < \kappa \rangle$ and $\langle p_\eta : \eta \in T \rangle$, where $T$ is the tree from Lemma 85 which exists as $\lambda^+ < 2^\lambda$, such that:

1. $M_i \in \mathbf{K}_\lambda$ for all $i < \kappa$;
2. $\langle M_i : i < \kappa \rangle$ is increasing and continuous;
3. For $\eta \leq \nu$, $p_\eta \leq p_\nu$;
4. For all $\eta$ of rank $i$ and $\nu_0 \neq \nu_1 \in T$, both of rank $i+1$ and extending $\eta$, $p_{\nu_0} \neq p_{\nu_1}$.
5. For all $i < \kappa$, $\langle p_{\eta \restriction \alpha} : \alpha < i \rangle$ is coherent.

Construction This is possible by induction on the rank of $\eta \in T$. At stage 0 let $p_{\langle\rangle}$ be any type (hence not minimal and having no minimal types above it). At successor stage, say the rank of $\eta$ is $\alpha+1$. Take extensions of $p_\eta$ to some extension in $\mathbf{K}_\lambda$ and there are enough distinct extensions by Fact 86. Use amalgamation and the extension property for types to ensure that they are over the same model. Without loss of generality assume that each $\eta$ has $\lambda$ extensions $\{\eta^i : i < \lambda\}$ at the next level. We find $\{p^i_\eta : i < \lambda\} \subseteq \mathbf{S}^{na}(N_\eta)$ distinct extensions of $p_\eta$ for each $\eta$. Amalgamate $N_\eta$ for all $\eta$ to obtain $M_{i+1}$ and $f^i_\eta : N_\eta \to M_{i+1}$. Extend each $f^i_\eta(p^i_\eta)$ to $p_{\eta^i} \in \mathbf{S}^{na}(M_{i+1})$. This finishes the successor case.

At limit stage take directed colimits.

Enough Take $M := \bigcup_{i<\kappa} M_i$, and $p_\eta$ be an upper bound of $\langle p_{\eta\restriction\alpha} : \alpha < \kappa\rangle$ for each branch (of length $\kappa$) of $T$. It is clear that $p_\eta \neq p_\nu$ if $\eta \neq \nu \in T$ by condition (4) of the construction. Therefore, $|\mathbf{S}(M)| \geq \lambda^{++}$. This is a contradiction as $\mathbf{K}$ is almost stable in $\lambda$ by Theorem 84. $\square$

**Corollary 90.** *Suppose $\lambda^+ < 2^\lambda < 2^{\lambda^+}$. Assume $\mathbf{K}$ is categorical in $\lambda$ and $\lambda^+$, and is almost stable in $\lambda$. If $\mathbf{K}_{\lambda^{++}} \neq \emptyset$, then $\mathbf{K}$ is stable in $\lambda$.*

In [She09a, §VI.4.2], it is shown that assuming $2^\lambda < 2^{\lambda^+}$, then one of three statements about $\lambda$ holds, which we denote by (A), (B) and (C). For our purpose, there is no need to present them explicitly.

**Fact 91.** *([She09a, §VI.4.5(4)]) Assume $2^\lambda < 2^{\lambda^+}$ and statement (A) holds for $\lambda$. If $\mathbf{K}$ has amalgamationin $\lambda$, $1 \leq \mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$, then for every non-algebraic type over any $M \in \mathbf{K}_\lambda$ there is a minimal type above it.*

**Fact 92.** *([She09a, §VI.4.9(2)]) Assume $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ and statement (B) or (C) holds for $\lambda$. If $\mathbf{K}$ has amalgamation in $\lambda$, is categorical in $\lambda^+$, $\mathbf{K}_{\lambda^{++}} \neq \emptyset$, and $|\mathbf{S}(N)| < 2^{\lambda^+}$ for the unique model in $\mathbf{K}_{\lambda^+}$, then for every non-algebraic type over any $M \in \mathbf{K}_\lambda$ there is a minimal type above it.*

**Lemma 93.** *Suppose that $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$. Assume $\mathbf{K}$ has amalgamation in $\lambda$ and is categorical in $\lambda^+$, and $|\mathbf{S}(N)| < 2^{\lambda^+}$ for the unique model $N \in \mathbf{K}_{\lambda^+}$. Then for every non-algebraic type over any $M \in \mathbf{K}_\lambda$ there is a minimal type above it.*

*Proof.* If there are no minimal types, one can keep extending a non-algebraic type (not minimal) to a model in $\lambda^+$. Since that extension is non-algebraic, with categoricity in $\lambda^+$ one can show that $\mathbf{K}_{\lambda^{++}} \neq \emptyset$. If statement (A) holds for $\lambda$, we use Fact 91. If (B) or (C) holds, and we use Fact 92. $\square$

**Theorem 94.** *Suppose that $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$. Assume $\mathbf{K}$ is categorical in $\lambda$ and $\lambda^+$, $\mathbf{K}_{\lambda^{++}} \neq \emptyset$, and $|\mathbf{S}(N)| < 2^{\lambda^+}$ for the unique model $N \in \mathbf{K}_{\lambda^+}$. Then $\mathbf{K}$ is stable in $\lambda$.*

*Proof.* By Fact 49 $\mathbf{K}$ has amalgamation in $\lambda$. By Lemma 93 there is a minimal type. This is enough as it implies the existence of an inevitable minimal type by Fact 87, which in turn implies that $\mathbf{K}$ is stable in $\lambda$ by Fact 88. □

We finish this section by observing that Fact 83 can be used to significantly simplify the proof of the following result due to Shelah.

**Fact 95.** *([She09a, VI.1.18, VI.1.20]) Assume $\mathbf{K}$ has amalgamation in $\lambda$. If $N \in \mathbf{K}_\lambda$, $\Gamma \subseteq \mathbf{S}^{na}(N)$ and $|\Gamma| > \lambda^+$. Then we can find $N^*$ and $\langle N_i : i < \lambda^{++}\rangle$ such that:*

1. *$N \leq_{\mathbf{K}} N^* <_{\mathbf{K}} N_i \in \mathbf{K}_\lambda$;*
2. *For all $i \neq j < \lambda^{++}$ and $c_i \in |N_i| - |N^*|$, $c_j \in |N_j| - |N^*|$, $\mathbf{gtp}(c_i/N^*, N_i) \neq \mathbf{gtp}(c_j/N^*, N_j)$;*
3. *there are $a_i \in |N_i|$ for $i < \lambda^{++}$ such that $\mathbf{gtp}(a_i/N, N_i) \in \Gamma$ is not realized in $N^*$, and these types are pairwise distinct; moreover $\mathbf{gtp}(a_i/N, N_i)$ is not realized in $N_j$ for $j < i$.*

*Proof.* Let $\mathbf{S}_*(M)$ be the set of non-algebraic extensions of $\Gamma$ over $M$ for $N \leq_{\mathbf{K}} M \in \mathbf{K}_\lambda$. Then for some $N \leq_{\mathbf{K}} N^* \in \mathbf{K}_\lambda$, there is no $\mathbf{S}_*$-inevitable set of types of size $\leq \lambda^+$; otherwise the assumptions[2] of Fact 83 holds. Now we build $N_i$ by induction. Let $N_0$ be such that $N^* \leq_{\mathbf{K}} N_0$, $\||N_0| - |N^*\|| = \lambda$, and there is $a_0 \in |N_0| - |N_i|$ realizing a type from $\mathbf{S}_*(N^*)$. Let a realization of this type be $a_0$. At stage $i$, choose $N_i$ such that $N_i$ omits $\bigcup_{j<i}\{\mathbf{gtp}(c/N^*, N_j) : c \in |N_j| - |N_*|\}$, $\||N_i| - |N^*\|| = \kappa_i$, and $N_i$ realizes a type from $\mathbf{S}_*(N^*)$. We can find such $N_i$ since the set is not $\mathbf{S}_*$-inevitable. Let a realization of this type be $a_i$. □

[2]The conditions here are slightly different, as we look at models $\mathbf{S}_*(M)$ for only $N \leq_{\mathbf{K}} M$ instead of every $M \in \mathbf{K}_\lambda$. However this is enough: one can check that the proof of Theorem 83 works.

## 4.3 Existence and categoricity above $\lambda^{++}$

### 4.3.1 Preliminaries

We introduce the basic properties of splitting we will use in this section. Recall that splitting for AECs was introduced in [She99, Definition 3.2].

**Definition 96.** *Let $M \in \mathbf{K}_\lambda$, $M \leq_{\mathbf{K}} N$ and $p \in \mathbf{S}(N)$ . $p$ ($\lambda$-)splits over $M$ if there are $N_1, N_2 \in \mathbf{K}_\lambda$ and $h : N_1 \cong_M N_2$ such that $M \leq_{\mathbf{K}} N_1, N_2 \leq_{\mathbf{K}} N$ and $h(p \restriction N_1) \neq p \restriction N_2$.*

We will use the following properties of non-splitting often in this section.

**Fact 97.** *Assume $\mathbf{K}$ has amalgamation and no maximal model in $\lambda$.*

1. *([Vas16b, 3.3]) Monotonicity: If $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2 \leq_{\mathbf{K}} M_3$, $p \in \mathbf{S}(M_3)$ does not split over $M_0$ and $M_0, M_1, M_2 \in \mathbf{K}_\lambda$, then $p \restriction M_2$ does not split over $M_1$.*
2. *Let $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$ all in $\mathbf{K}_\lambda$ and $M_1$ is universal over $M_0$.*
   - *([Van06, I.4.10]) Weak extension: If $p \in \mathbf{S}^{na}(M_1)$ does not split over $M_0$, then there is $q \in \mathbf{S}^{na}(M_2)$ such that $q$ extends $p$ and $q$ does not split over $M_0$.*
   - *([Van06, I.4.12]) Weak uniqueness: If $p, q \in \mathbf{S}(M_2)$, $p \restriction M_1 = q \restriction M_1$, and $p, q$ do not split over $M_0$, then $p = q$.*
3. *([Vas16b, 3.7]) Weak transitivity: If $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_1' \leq_{\mathbf{K}} M_2$ all in $\mathbf{K}_\lambda$, $M_1'$ universal over $M_1$ and $p \in \mathbf{S}^{na}(M_2)$ such that $p$ does not split over $M_1$ and $p \restriction M_1'$ does not split over $M_0$, then $p$ does not split over $M_0$.*

**Fact 98** ([She99, 3.3], [SV99, Theorem 2.2.1] )**.** *(Weak universal local character) Assume $\mathbf{K}$ has amalgamation, no maximal model and is stable in $\lambda$. If $\langle M_i : i \leq \lambda \rangle$ is an increasing continuous chain in $\mathbf{K}_\lambda$ with $M_{i+1}$ universal over $M_i$ for all $i < \lambda$ and $p \in \mathbf{S}^{na}(M_\lambda)$, then there is $i < \lambda$ such that $p \restriction M_{i+1}$ does not split over $M_i$.*

### 4.3.2 Existence of a model in $\lambda^{++}$

We focus first on the existence of a model of cardinality $\lambda^{++}$.

**Definition 99.** *Assume $\mathbf{K}$ is stable in $\lambda$. Splitting is* continuous in $\lambda$ *if for any limit ordinal $\delta < \lambda^+$ and any increasing continuous chain $\langle M_i : i \leq \delta \rangle$ with $M_{i+1}$ universal over $M_i$ for all $i < \delta$, if $p \in \mathbf{S}(M_\delta)$ is such that $p \restriction M_i$ does not split over $M_0$ for all $i < \delta$, then $p$ does not split over $M_0$.*

The following result is folklore, but we provide a proof as we could not find a reference.

**Lemma 100.** *Assume* $\mathbf{K}$ *has amalgamation, no maximal model and is stable in* $\lambda$*. If* $\mathbf{K}$ *is* $(<\lambda^+, \lambda)$*-local, then splitting is continuous in* $\lambda$*.*

*Proof.* Let $\delta < \lambda^+$ be a limit ordinal, $\langle M_i : i \leq \delta \rangle$ be an increasing continuous chain with $M_{i+1}$ universal over $M_i$ for all $i < \delta$ and $p \in \mathbf{S}(M_\delta)$ such that $p \upharpoonright M_i$ does not split over $M_0$ for all $i < \delta$.

Applying the weak extension property to $p \upharpoonright M_1$, there is $q \in \mathbf{S}^{na}(M_\delta)$ such that $q$ extends $p \upharpoonright M_1$ and $q$ does not split over $M_0$. We show that $p = q$. Since $\mathbf{K}$ is $(<\lambda^+, \lambda)$-local, it is enough to show that $p \upharpoonright M_i = q \upharpoonright M_i$ for all $i < \delta$.

Let $i < \delta$. When $i = 0$ or $i = 1$ the result is clear as $p \upharpoonright M_1 \leq q$. When $i > 1$ the result follows from weak uniqueness and the fact that $q \upharpoonright M_i$ does not split over $M_0$ by monotonicity. □

**Remark 101.** *Continuity of splitting also follows from* $\lambda$*-superstability which can be derived from categoricity, amalgamation, and arbitrarily large models [She09a, §II], [GV05, 2.9].*

**Lemma 102.** *Let* $\mathbf{K}$ *be an AEC and let* $\lambda \geq \mathrm{LS}(\mathbf{K})$*. If* $\mathbf{K}$ *has amalgamation, no maximal model and is stable in* $\lambda$*, and splitting is continuous in* $\lambda$*, then* $\mathbf{K}$ *has a model in* $\lambda^{++}$*.*

*Proof.* We show $\mathbf{K}$ has no maximal models in $\lambda^+$. Assume for the sake of contradiction that there is $N \in \mathbf{K}_{\lambda^+}$ a maximal model.

First build a strictly increasing continuous chain $\langle M_i : i \leq \lambda \rangle$ in $\mathbf{K}_\lambda$ with $M_{i+1}$ universal over $M_i$ for all $i < \lambda$ and $M_i \leq_{\mathbf{K}} N$ for every $i < \lambda$. This is possible by stability and amalgamation in $\lambda$ and the maximality of $N$. Pick $p \in \mathbf{S}^{na}(M_\lambda)$. It follows from Fact 98, that there exists $i < \lambda$ such that $p \upharpoonright M_{i+1}$ does not split over $M_i$.

Let $\{n_i : i < \lambda^+\}$ be an enumeration of $N$ and $N_* = M_i$. We build an increasing continuous chain $\langle N_i : i < \lambda^+ \rangle$ in $\mathbf{K}_\lambda$ and $\langle p_i : i < \lambda^+ \rangle$ a chain of types such that:

1. $N_0 = M_{i+1}$ and $p_0 = p \upharpoonright M_{i+1}$;
2. for every $i < \lambda^+$, $n_i \in N_{i+1}$, $N_i \leq_{\mathbf{K}} N$ and $N_{i+1}$ is universal over $N_i$;
3. for every $i < \lambda^+$, $p_i \in \mathbf{S}^{na}(N_i)$ does not split over $N_*$;
4. if $i < j < \lambda^+$, then $p_i \leq p_j$;

5. for every $j < \lambda^+$, $\langle p_i : i < j\rangle$ is coherent .

Construction The base step is given by Condition (1) and for $i$ limit, the construction can be carried out by coherence of the sequence and the fact that splitting is continuous in $\lambda$ by assumption. So we do the case when $i = j + 1$. Let $L$ be the structure obtained by applying the Löwenheim-Skolem-Tarski axiom to $N_j \cup \{n_j\}$ in $N$ and $L^* \in \mathbf{K}_\lambda$ a universal model over $L$, $L^*$ exists by stability and amalgamation in $\lambda$. Using amalgamation in $\lambda$ and the maximality of $N$ there is $f : L^* \underset{L}{\rightarrow} N$. Let $N_{j+1} = f[L^*]$. As $N_{j+1}$ is universal over $N_*$, applying the weak extension property to $p_j$ one obtains $p_{j+1} \in \mathbf{S}^{na}(N_{j+1})$ extending $p_j$ and such that $p_{j+1}$ does not split over $N_*$. It is easy to check that $N_{j+1}$ and $p_{j+1}$ satisfies Conditions (2) to (5).

Enough Let $p^* \in \mathbf{S}^{na}(\bigcup_{i<\lambda^+} N_i)$ be an upper bound of the coherent sequence $\langle p_i : i < \lambda^+\rangle$. Since $p^*$ is not algebraic by Remark 32 and $N = \bigcup_{i<\lambda^+} N_i$ by Condition (2) of the construction, it follows that $N$ has a proper extension. This contradicts the assumption that $N$ was a maximal model. □

We are ready to prove the main equivalence of the chapter.

**Theorem 103.** *Suppose $\lambda^+ < 2^\lambda < 2^{\lambda^+}$. Assume $\mathbf{K}$ is categorical in $\lambda$ and $\lambda^+$, $\mathbf{K}$ is $(< \lambda^+, \lambda)$-local and $\mathbf{K}$ is almost stable in $\lambda$. The following are equivalent.*

1. *$\mathbf{K}$ has a model in $\lambda^{++}$.*
2. *$\mathbf{K}$ is stable in $\lambda$.*

*Proof.* (1) ⇒ (2): Corollary 90.

(2) ⇒ (1): $\mathbf{K}$ has amalgamation in $\lambda$ by Fact 49, no maximal models in $\lambda$ by categoricity in $\lambda$ and $\mathbf{K}_{\lambda^+} \neq \emptyset$ and stable in $\lambda$ by assumption. Moreover, splitting is continuous in $\lambda$ by Lemma 100. Therefore $\mathbf{K}$ has a model in $\lambda^{++}$ by Lemma 102. □

### 4.3.3 Categoricity above $\lambda^{++}$

We show how to transfer categoricity. A key assumption we will use to transfer categoricity that we did not have in the previous section is tameness.

The following two results are known, but we could not find a reference so we sketch the proof for the convinience of the reader.

**Fact 104.** *If* $\mathbf{K}$ *has amalgamation in* $\lambda$*,* $\mathbf{K}$ *is stable in* $\lambda$*, and* $\mathbf{K}$ *is categorical in* $\lambda^+$*, then the model of cardinality* $\lambda^+$ *is* $\lambda^+$*-model-homogeneous above* $\lambda$*.*

*Proof.* We can assume without loss of generality that $\mathbf{K}$ has joint embedding and no maximal models in $\lambda$. If not, partition $\mathbf{K}_\lambda$ into equivalence classes given by $M$ is equivalent to $N$ if they can be $\mathbf{K}$-embedded into a model in $\mathbf{K}_\lambda$, and restrict yourself to the class that generates the model in $\lambda^+$.

Build a strictly increasing continuous chain $\langle M_i : i < \lambda^+ \rangle$ in $\mathbf{K}_\lambda$ with $M_{i+1}$ universal over $M_i$ for all $i < \lambda$. This is possible by stability, joint embedding, no maximal, and amalgamation in $\lambda$. Let $M_{\lambda^+} = \bigcup_{i<\lambda^+} M_i \in \mathbf{K}_{\lambda^+}$. Using a cofinality argument, it is clear that $M_{\lambda^+}$ is model-homogeneous above $\lambda$. Therefore, the model of cardinality $\lambda^+$ is $\lambda^+$-model-homogeneous above $\lambda$. □

**Fact 105.** *Assume* $\mathbf{K}$ *has amalgamation, no maximal model and is stable in* $\lambda$ *and* $\mathbf{K}$ *is categorical in* $\lambda^+$*. If* $N \in \mathbf{K}_{\lambda^+}$ *and* $p \in \mathbf{S}(N)$*, then there is* $M \in \mathbf{K}_\lambda$ *such that* $M \leq_{\mathbf{K}} N$ *and* $p$ *does not split over* $M$*.*

*Proof.* It follows from Fact 98 using that $N$ is homogeneous above $\lambda$ by Fact 104 and an analogous argument to that of [Bal09, 12.5]. □

The following weakening of amalgamation was isolated in [Vas17a, 4.11] and developed in [Vas17a, §4]. Universal classes and classes with intersections have weak amalgamation.

**Definition 106.** $\mathbf{K}$ *has* weak amalgamation *if whenever* $\mathbf{gtp}(a_1/M, N_1) = \mathbf{gtp}(a_2/M, N_2)$ *there are* $N_1' \leq_{\mathbf{K}} N_1$ *and* $N_2 \leq_{\mathbf{K}} N_3$ *such that* $\{a_1\} \cup M \subseteq N_1'$ *and* $f : N_1' \underset{M}{\longrightarrow} N_3$ *is a* $\mathbf{K}$*-embedding with* $f(a_1) = a_2$*.*

**Lemma 107.** *Let* $\mathbf{K}$ *be an AEC and let* $\lambda \geq \mathrm{LS}(\mathbf{K})$*. Assume* $\mathbf{K}$ *has amalgamation in* $\lambda$*,* $\mathbf{K}$ *is stable in* $\lambda$*, and splitting is continuous in* $\lambda$*. If* $\mathbf{K}$ *is categorical in* $\lambda^+$*,* $\mathbf{K}$ *is* $\lambda$*-tame and has weak amalgamation, then* $\mathbf{K}_{\geq\lambda}$ *has amalgamation and* $\mathbf{K}$ *has arbitrarily large models.*

*Proof.* As before we can assume without loss of generality that $\mathbf{K}$ has joint embedding and no maximal models in $\lambda$. Moreover, we assume that $\mathbf{K}_{<\lambda} = \emptyset$.

Let $\mathfrak{s} = (\mathbf{K}, \mathop{\smile\!\!\!\!|}, \mathbf{S}^{bs})$ be given by:

- For $M \in \mathbf{K}_{\lambda^+}$, $\mathbf{S}^{bs}(M) = \mathbf{S}^{na}(M)$.

- For $M, N, R \in \mathbf{K}_{\lambda^+}$ we define: $a \mathop{\smile\!\!\!\!\!\!|}\nolimits_M^R N$ if and only if $M \leq_{\mathbf{K}} N \leq_{\mathbf{K}} R$, $a \in |R| \backslash |N|$ and there is $M' \in \mathbf{K}_\lambda$ with $M' \leq_{\mathbf{K}} M$ such that for every $N' \in \mathbf{K}_\lambda$ with $M' \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N$ there is $M_0' \in \mathbf{K}_\lambda$ such that $M_0' \leq_{\mathbf{K}} M'$, $M'$ is universal over $M_0'$ and $\mathbf{gtp}(a/N, R) \restriction N'$ does not split over $M_0'$. We say that $\mathbf{gtp}(a/N, R)$ does not $\lambda^+$-fork over $M$.

<u>Claim</u>: $\mathfrak{s} = (\mathbf{K}, \mathop{\smile\!\!\!\!\!\!|}, \mathbf{S}^{bs})$ is a w-good $\lambda^+$-frame with density.

<u>Proof of Claim</u>: This frame was first considered in [Vas16b, 4.2,3.8] under different assumptions, we show that everything still goes through in our setting. First observe that all the models in $\lambda^+$ are $\lambda^+$-model-homogeneous by Fact 104 so we can apply the results of [Vas16b, §4, 5]. $\mathfrak{s}$ is a pre-$\lambda^+$-frame by [Vas16b, 4.6], $\mathbf{K}_{\lambda^+}$ has no maximal models by Lemma 102 and has joint embedding by categoricity in $\lambda^+$, $\mathfrak{s}$ has: density by [Vas16b, 4.9], uniqueness by $\lambda$-tameness and [Vas16b, 5.3], and transitivity by [Vas16b, 4.10]. We show continuity, existence of non-forking extensions, and amalgamation in $\lambda^+$ as these are shown in [Vas16b] under additional assumptions.

- <u>Continuity</u>: Let $\delta < \lambda^{++}$ be a limit ordinal which we may assume to be a regular cardinal, $\langle M_i : i < \delta \rangle$ be an increasing continuous chain in $\mathbf{K}_{\lambda^+}$ and $p \in \mathbf{S}^{na}(M_\delta)$ such that for every $i < \delta$, $p \restriction M_i$ does not $\lambda^+$-fork over $M_0$. There is $M^* \in \mathbf{K}_\lambda$ such that $M^* \leq_{\mathbf{K}} M_\delta$ and $p$ does not split over $M^*$ by Fact 105. There are two cases to consider:

  <u>Case 1</u>: $\delta = \lambda^+$. Then there is $i < \lambda^+$, such that $M^* \leq_{\mathbf{K}} M_i$. Hence $p$ does not $\lambda^+$-fork over $M_i$ by [Vas16b, 4.8]. Then by the assumption that $p \restriction M_i$ does not $\lambda^+$-fork over $M_0$ and transitivity of $\mathfrak{s}$, we have that $p$ does not $\lambda^+$-fork over $M_0$.

  <u>Case 2</u>: $\delta \leq \lambda$. For each $i < \delta$, there is $M_0^i \in \mathbf{K}_\lambda$ such that $M_0^i \leq_{\mathbf{K}} M_0$ and $p \restriction M_i$ does not split over $M_0^i$ by [Vas16b, 4.8]. Since $\delta \leq \lambda$, using stability in $\lambda$, monotonicity of splitting and that $M_0$ is $\lambda^+$-model-homogeneous, there are $M_{0,0} \leq_{\mathbf{K}} M_{0,1} \in \mathbf{K}_\lambda$ such that $M_{0,1} \leq_{\mathbf{K}} M_0$, $M_{0,1}$ is universal over $M_{0,0}$ and for every $i < \delta$, $p \restriction M_i$ does not split over $M_{0,0}$.

  Let $N^*, N^{**} \leq_{\mathbf{K}} M_\delta$ both in $\mathbf{K}_\lambda$ such that $N^{**}$ is universal over $N^*$ and $M_{0,0} \cup M^* \subseteq N^*$. Using stability in $\lambda$, monotonicity of splitting and that the $M_i$'s are $\lambda^+$-model-homogeneous, one can build $\langle N_i : i \leq \delta \rangle$ in $\mathbf{K}_\lambda$ increasing continuous such that $N_0 = M_{0,1}$, $N_{i+1}$ is universal over $N_i$, $N_i \leq_{\mathbf{K}} M_i$, $N^{**} \cap M_{i+1} \subseteq N_{i+1}$ and $p \restriction N_i$ does not split over $M_{0,0}$. Since splitting is continuous in $\lambda$ by assumption, $p \restriction N_\delta$ does not split over $M_{0,0}$.

We show that $M_{0,1} \leq_{\mathbf{K}} M_0$ witnesses that $p$ does not $\lambda^+$-fork over $M_0$. Let $N' \in \mathbf{K}_\lambda$ with $M_{0,1} \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} M_\delta$ and $M_0' = M_{0,0}$, we show that $p \restriction N'$ does not split over $M_{0,0}$. Let $L \in \mathbf{K}_\lambda$ be the structure obtained by applying the Löwenheim-Skolem-Tarski axiom to $N_\delta \cup N^{**} \cup N'$ in $M_\delta$. By monotonicity of splitting $p \restriction L$ does not split over $N^*$. Since $p \restriction N_\delta$ does not split over $M_{0,0}$ and $N_\delta$ is universal over $N^*$ because $N^{**} \leq_{\mathbf{K}} N_\delta$, then $p \restriction L$ does not split over $M_{0,0}$ by weak transitivity. Therefore $p \restriction N'$ does not split over $M_{0,0}$ by monotonicity of splitting.

- Existence of non-forking extension: Let $M \leq_{\mathbf{K}} N$ both in $\mathbf{K}_{\lambda^+}$ and $p \in \mathbf{S}^{na}(M)$. There is $M^* \in \mathbf{K}_\lambda$ such that $M^* \leq_{\mathbf{K}} M$ and $p$ does not split over $M^*$ by Fact 105. First build an increasing continuous chain $\langle M_i : i < \lambda^+ \rangle$ in $\mathbf{K}_\lambda$ with $M_i \leq_{\mathbf{K}} M$ for all $i < \lambda^+$, $M_0$ is universal over $M^*$ and $M = \bigcup_{i<\lambda^+} M_i$ .

  Let $\{n_i : i < \lambda^+\}$ be an enumeration of $N$. We build, as in Lemma 102 using that $N$ is $\lambda^+$-model-homogeneous, an increasing continuous chain $\langle N_i : i < \lambda^+ \rangle$ in $\mathbf{K}_\lambda$ and $\langle p_i : i < \lambda^+ \rangle$ a chain of types such that:

  1. $N_0 = M_0$ and $p_0 = p \restriction M_0$;
  2. for every $i < \lambda^+$, $n_i \in N_{i+1}$, $N_i \leq_{\mathbf{K}} N$ and $N_{i+1}$ is universal over $N_i$;
  3. for every $i < \lambda^+$, $p_i \in \mathbf{S}^{na}(N_i)$ does not split over $M^*$;
  4. if $i < j < \lambda^+$, then $p_i \leq p_j$;
  5. for every $j < \lambda^+$, $\langle p_i : i < j \rangle$ is coherent .

  Let $p_{\lambda^+} \in \mathbf{S}^{na}(\bigcup_{i<\lambda^+} N_i) = \mathbf{S}^{na}(N)$ be an upper bound of the coherent sequence $\langle p_i : i < \lambda^+ \rangle$. We show that $p_{\lambda^+} \geq p$ and that $p_{\lambda^+}$ does not $\lambda^+$-fork over $M$.

  We show that for every $i < \lambda^+$, $p_i \restriction M_i = p \restriction M_i$. This is enough to show that $p_{\lambda^+} \geq p$ as $\mathbf{K}$ is $\lambda$-tame. Let $i < \lambda^+$. Observe $p_i \restriction M_i$ does not split over $M^*$ by Condition (3), $p \restriction M_i$ does not split over $M^*$ by monotonicity of splitting, $(p_i \restriction M_i) \restriction M_0 = p_0 = p \restriction M_0 = (p \restriction M_i) \restriction M_0$ by Conditions (1), (4) and $M_0$ is universal over $M^*$, then $p_i \restriction M_i = p \restriction M_i$ by weak uniqueness.

  We show that $M_0 \leq_{\mathbf{K}} M$ witnesses that $p$ does not $\lambda^+$-fork over $M$. Let $N' \in \mathbf{K}_\lambda$ with $M_0 \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N$ and $M_0' = M^*$. Observe that $p \restriction N'$ does not split over $M^*$ by Condition (3) and monotonicity of splitting.

- Amalgamation in $\lambda^+$: It follows from density and existence of non-forking extension of $\mathfrak{s}$ and weak amalgamation by [Vas17a, 4.16]. $\dagger_{\text{Claim}}$

Since $\mathbf{K}$ has weak amalgamation, is $\lambda$-tame and $\mathfrak{s}$ is a w-good $\lambda$-frame with density, one can show that $\mathbf{K}$ has a $[\lambda^+, \infty)$-w-good frame with density following the arguments of [Vas17a, 4.16] and [MA20, 3.24] (see also [Bon13]). In particular, $\mathbf{K}_{\geq\lambda}$ has amalgamation and $\mathbf{K}$ has arbitrarily large models. $\square$

**Theorem 108.** *Let $\mathbf{K}$ be an AEC with weak amalgamation and let $\lambda \geq \mathrm{LS}(\mathbf{K})$ be such that $\mathbf{K}$ is $\lambda$-tame. Assume $\mathbf{K}$ has amalgamation in $\lambda$, $\mathbf{K}$ is stable in $\lambda$, and splitting is continuous in $\lambda$. If $\mathbf{K}$ is categorical in $\lambda$ and $\lambda^+$, then $\mathbf{K}$ is categorical in all $\mu \geq \lambda$.*

*Proof.* $\mathbf{K}_{\geq\lambda}$ has amalgamation and $\mathbf{K}$ has arbitrarily large models by Lemma 107. Therefore, $\mathbf{K}$ is categorical in all $\mu \geq \lambda$ by Grossberg-VanDieren theorem [GV06, 5.2, 6.3]. $\square$

A simpler result to state is the following. The result directly follows from the previous theorem as universal classes are $(< \aleph_0)$-tame [Vas17a, 3.7] and have weak amalgamation.

**Corollary 109.** *Let $\mathbf{K}$ be a universal class and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. Assume $\mathbf{K}$ has amalgamation in $\lambda$ and $\mathbf{K}$ is stable in $\lambda$. If $\mathbf{K}$ is categorical in $\lambda$ and $\lambda^+$, then $\mathbf{K}$ is categorical in all $\mu \geq \lambda$.*

*Proof.* Universal classes are $(< \aleph_0)$-tame [Vas17a, 3.7]. Therefore, they are $(< \lambda^+, \lambda)$-local and $\lambda$-tame (see for example [MAY24, 2.6]). Then splitting is continuous by Lemma 100. As universal classes have weak amalgamation, the result follows from Theorem 108. $\square$

Using the results of the previous two sections we have the following variation of the previous result.

**Corollary 110.** *Assume $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$. Let $\mathbf{K}$ be a universal class and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. If $\mathbf{K}$ is categorical in $\lambda$, $\lambda^+$, $\mathbf{K}_{\lambda^{++}} \neq \emptyset$ and $|\mathbf{S}(N)| < 2^{\lambda^+}$ for the unique $N \in \mathbf{K}_{\lambda^+}$, then $\mathbf{K}$ is categorical in all $\mu \geq \lambda$.*

*Proof.* The result follows from Theorem 94 and the previous corollary. $\square$

# Chapter 5

# An analogue of NIP in AECs

## 5.1 Introduction

This section is essentially the content of [Yan23]. NIP theories, also called dependent theories, were first discovered by Shelah in [She71b].

**Notation 111.** *For any formula $\varphi$ and a condition $i$, $\varphi^i$ means $\varphi$ itself when $i$ holds, and $\neg\varphi$ otherwise.*

**Definition 112.** *Let $T$ be a complete theory in first order logic and $\phi(\bar{x}, \bar{y})$ a formula in its language. We say $\phi$ has the* independence property *if for all $n < \omega$,*

$$T \models \exists \bar{x}_1 \dots \exists \bar{x}_n \bigwedge_{w \subseteq n} \bigwedge_{i<n} \exists \bar{y}_w \phi(\bar{x}_i, \bar{y}_w)^{i \in w}.$$

Roughly speaking, $\phi(\bar{x}_i, \bar{y}_w)$ behaves like "$i \in w$" for $i < n$ and $w \subseteq n$. In this situation we say $T$ is able to *encode subsets of* $n$ for all $n < \omega$. Any easy application of the compactness theorem gives another equivalent characterization of the independence property.

**Fact 113.** *Let $T$ and $\phi$ as in the previous definition. $\phi$ has the independence property if and only if for some $M \models T$ and $\{a_i : i < \omega\} \subseteq |M|$, for all $S \subseteq \omega$ there is $\bar{b}_S$ such that*

$$M \models \phi(\bar{a}_i, \bar{b}_S) \iff i \in S.$$

That is, $\phi$ (or $T$) is able to encode subsets of $\omega$. In fact, by the compactness theorem again, one can replace $\omega$ with any infinite cardinal, so $T$ is able to encode subsets of any cardinal.

**Definition 114.** *We say $T$ has the* independence property *if for some $\phi$, one of the equivalent conditions above holds. We say $T$ is* dependent *or* NIP, *when $T$ does not have the independence property.*

As stability is just a combinatorial condition on the number of types, it has a relatively obvious generalization to abstract elementary classes. For the independence property it is no so clear since there are no formulas to work with. Fortunately there is an equivalent condition to the independence property, under additional cardinal arithmetic assumptions.

**Definition 115.** *A* tree $(T,<)$ *is a partially ordered set such that for all $t \in T$, the set $\{s \in T : s < T\}$ is a well ordering.*

**Definition 116.** *Let $(T,<)$ be a tree.*

1. *The elements of $T$ are sometimes called* nodes.
2. *For all $t \in T$, the order type of $\{s \in T : s < t\}$ is called the rank or height of $t$.*
3. *The $\alpha$-th level of $T$ is the set of nodes that are of rank $\alpha$.*
4. *A* branch *of $T$ is a maximal linearly ordered subset of $T$.*
5. *The* length *of a branch is its order type.*

**Definition 117.** *[She71b, CKS16] For a cardinal $\lambda$,*

$$\mathrm{ded}\,\lambda := \sup\{\kappa \mid \exists \text{ a linear ordering } I \text{ of cardinality } \kappa \text{ and a dense subset of cardinality } \leq \lambda\}.$$

This notion was first discussed in [Bau76]. The notation ded is intended for "Dedekind cuts". There are two conventions of defining this cardinal. [She78] and [She71b] uses the other, where it is defined to be the least cardinal $\mu$ such that no linear ordering of cardinality $\mu$ has a dense subset of cardinality $\leq \lambda$. The sup in the definition of ded is attained if and only if the two conventions agree.

There are multiple equivalent definitions of ded $\lambda$.

**Fact 118.** *[CS16, CKS16] The following are equal:*

1. *$\mathrm{ded}\,\lambda$;*
2. $\sup\{\kappa \mid \exists$ *a regular $\mu$ and a tree $T$ with $\leq \lambda$ nodes and $\kappa$ branches of length $\mu$*, $|T| = \kappa\}$.
3. $\sup\{\kappa \mid \exists$ *a tree $T$ with $\leq \lambda$ nodes and $\kappa$ branches*$\}$.

**Fact 119.**
1. *$\lambda < \mathrm{ded}\,\lambda$. Consider the tree $({}^{<\mu}2, \subseteq)$ where $\mu$ is the least such that $2^\mu > \lambda$. ${}^{\mu}2$ is exactly the set of branches.*

2. *ded* $\lambda \leq 2^\lambda$. *Any tree of size* $\lambda$ *has at most* $2^\lambda$ *subsets, and every branch must be a subset.*

**Fact 120.** *[She78, II.4.11] Let* $T$ *be a complete first order theory and* $\phi$ *a formula in its language.* $\lambda$ *is an infinite cardinal such that* $2^\lambda > \text{ded } \lambda$. *The following are equivalent:*

1. $\phi$ *has the independence property.*
2. $|\mathbf{S}_\phi(A)| > \text{ded } |A|$ *for some infinite set* $A$, $|A| = \lambda$.
3. $|\mathbf{S}_\phi(A)| = 2^{|A|}$ *for some infinite set* $A$, $|A| = \lambda$.

**Fact 121.** *[She78, II.4.12] Let* $T$ *be a complete theory in countable language, and* $f_T(\lambda) := \sup\{|\mathbf{S}(M)| \mid M \models T, \|M\| = \lambda\}$. *Then* $f_T(\lambda)$ *is exactly one of:* $\lambda$, $\lambda + 2^{\aleph_0}$, $\lambda^{\aleph_0}$, *ded* $\lambda$, $(\text{ded } \lambda)^{\aleph_0}$ *or* $2^\lambda$. *See also [Kei76].*

It is reasonable to propose the following definition:

**Definition 122.** *Let* $\mathbf{K}$ *be an AEC,* $\lambda \geq LS(\mathbf{K})$. $\mathbf{K}_\lambda$ *has NIP if for all* $M \in \mathbf{K}_\lambda$, $|\mathbf{S}(M)| \leq \text{ded } \lambda$.

At present it is unclear that we have discovered the "correct" notion. In fact, it is plausible that there are several different notions that are equivalent when $\mathbf{K}$ is an elementary class, but distinct for some non-elementary $\mathbf{K}$. One weakness of our definition is that unlike the corresponding first order notion, it is probably not absolute.

Grossberg raised the following question:

**Question 123.** *Is there an equivalent notion which does not rely on extra set theoretic assumptions. (at least for AECs* $\mathbf{K}$ *with* $LS(\mathbf{K}) = \aleph_0$ *which are also* $PC_{\aleph_0}$*)?*

The following are two examples of an abstract elementary class satisfying NIP that is not elementary or stable.

**Fact 124.** *[JS13, 2.5.8] Assume* $\mathbf{K}_\lambda$ *has joint embedding, no maximal model and amalgamation. Suppose there is* $\mathbf{S}^{bs} \subseteq \mathbf{S}^{na}$ *family of types on* $K$ *satisfying only (Density), (Invariance), and for all* $M \in \mathbf{K}_\lambda$, $|\mathbf{S}^{bs}(M)| \leq \lambda^+$. *See Definitions 127 and 129.*

1. *If* $\langle M_\alpha \in \mathbf{K}_\lambda \mid \alpha < \lambda^+ \rangle$ *is increasing and continuous, and there is a stationary set* $S \subseteq \lambda^+$ *such that for every* $\alpha \in S$ *and every model* $N$, *with* $M_\alpha \leq_{\mathbf{K}} N$, *there is a type* $p \in \mathbf{S}^{bs}(M_\alpha)$ *which is realized in* $M_{\lambda^+} := \bigcup_{i<\lambda^+} M_i$ *and in* $N$, *then* $M_{\lambda^+}$ *is saturated in* $\lambda^+$ *above* $\lambda$.
2. *For all* $M \in \mathbf{K}_\lambda$, $|\mathbf{S}(M)| \leq \lambda^+$.

**Example 125.** *[JS13, 2.2.4] Let $\lambda$ be a cardinal. Let $P$ be a family of $\lambda^+$ subsets of $\lambda$. Let $\tau := \{R_\alpha : \alpha < \lambda\}$ where each $R_\alpha$ is an unary predicate. Let $\mathbf{K}$ be the class of models $M$ for $\tau$ such that for each $a \in |M|$, $\{\alpha \in \lambda \mid M \models R_\alpha(a)\} \in P$. Note that $\mathbf{K}$ is not elementary. Let $\leq_{\mathbf{K}}$ be the substructure relation on $\mathbf{K}$. The trivial $\lambda$-frame on $\mathbf{K}_\lambda$ satisfies the axioms of a semi-good $\lambda$-frame[JS13, 2.1.3], so in particular by Fact 124 $\mathbf{K}_\lambda$ satisfies NIP. On the other hand, it is unstable.*

The next is an algebraic example of an abstract elementary class that satisfies NIP and is not elementary or stable.

**Example 126.** *(ded $\lambda = (\text{ded } \lambda)^{\aleph_0}$) Let $\mathbf{K}$ be the class of real closed fields, and $F \leq_{\mathbf{K}} L$ if and only if $F \preceq L$ and $L/F$ is a normal extension, so $(\mathbf{K}, \leq_{\mathbf{K}})$ is not elementary. Since $(\mathbf{K}, \preceq)$ is NIP but unstable, the number of $\mathbb{L}_{\omega,\omega}$ syntactic types over $M \in \mathbf{K}_\lambda$, which are orbits of $Aut_M(\mathfrak{C})$, coincide with Galois types $\mathbf{S}(M)$. The number of types is bounded by ded $\lambda = (\text{ded } \lambda)^{\aleph_0}$ but strictly more than $\lambda$.*

## 5.2 The w*-good frame

In this section we define w*-good frames, and show that $\mathbf{K}_\lambda$ has NIP if and only if $\mathbf{K}$ has a w*-good $\lambda$-frame under additional assumptions. Mazari-Armida [MA20] introduced the w-good frame which is weaker than all other such notions. The w*-frame is similar in this sense, but probably incomparable to the w-good frame.

Then we show that the negation of NIP implies that the class is enable to encode subsets and compute the Hanf number of this property.

**Definition 127.** *[She09a, §III.0] Let $\lambda < \mu$, where $\lambda$ is a cardinal, and $\mu$ is a cardinal or $\infty$. A* pre-$[\lambda, \mu)$-frame *is a triple $\mathfrak{s} = (\mathbf{K}, \mathop{\smile}\limits^{\vert}, S^{bs})$ such that:*

1. *$\mathbf{K}$ is an AEC with $\lambda \geq LS(\mathbf{K})$ and $\mathbf{K}_\lambda \neq \emptyset$.*
2. *$\mathbf{S}^{bs} \subseteq \bigcup_{M \in \mathbf{K}_{[\lambda,\mu)}} \mathbf{S}^{na}(M)$. Let $\mathbf{S}^{bs}(M) := \mathbf{S}(M) \cap \mathbf{S}^{bs}$. Types in this family are called* basic types.
3. *$\mathop{\smile}\limits^{\vert}$ is a relation on quadruples $(M_0, M_1, a, N)$, where $M_0 \leq_{\mathbf{K}} M_1 \leq N$, $a \in |N|$ and $M_0, M_1, N \in \mathbf{K}_{[\lambda,\mu)}$. We write $a \mathop{\smile}\limits^{\vert}{}^{N}_{M_0} M_1$, or we say $\mathbf{gtp}(a/M_1, N)$ does not fork over $M_0$ when the relation $\mathop{\smile}\limits^{\vert}$ holds for $(M_0, M_1, a, N)$.*

4. *(Invariance) If* $f : N \cong N'$ *and* $a \underset{M_0}{\overset{N}{\downarrow}} M_1$*, then* $f(a) \underset{f[M_0]}{\overset{N'}{\downarrow}} f[M_1]$*. If* $\mathbf{gtp}(a/M_1, N) \in \mathbf{S}^{bs}(M_1)$*, then* $\mathbf{gtp}(f(a)/f[M_1], N') \in \mathbf{S}^{bs}(f[M_1])$*.*
5. *(Monotonicity) If* $a \underset{M_0}{\overset{N}{\downarrow}} M_1$ *and* $M_0 \leq_{\mathbf{K}} M_0' \leq_{\mathbf{K}} M_1' \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N \leq_{\mathbf{K}} N''$ *with* $N'' \in \mathbf{K}_{[\lambda,\mu)}$ *and* $a \in |N'|$*, then* $a \underset{M_0'}{\overset{N'}{\downarrow}} M_1'$ *and* $a \underset{M_0'}{\overset{N''}{\downarrow}} M_1'$*.*
6. *(Non-forking Types are Basic) If* $a \underset{M}{\overset{N}{\downarrow}} M$ *then* $\mathbf{gtp}(a/M, N) \in \mathbf{S}^{bs}(M)$*.*

**Definition 128.** *[MA20, 3.6] A pre-*$[\lambda, \mu)$*-frame* $\mathfrak{s} = (\mathbf{K}, \downarrow, \mathbf{S}^{bs})$ *is a* w-good frame *if:*

1. $\mathbf{K}_{[\lambda,\mu)}$ *has amalgamation, joint embedding and no maximal model.*
2. *(Weak Density) For all* $M <_{\mathbf{K}} N \in \mathbf{K}_\lambda$*, there is* $a \in |N| - |M|$ *and* $M' \leq N' \in \mathbf{K}_{[\lambda,\mu)}$ *such that* $(a, M, N) \leq (a, M', N')$ *and* $\mathbf{gtp}(a/M', N') \in \mathbf{S}^{bs}(M')$*.*
3. *(Existence of Non-Forking Extension) If* $p \in \mathbf{S}^{bs}(M)$ *and* $M \leq_{\mathbf{K}} N$*, then there is* $q \in \mathbf{S}^{bs}(N)$ *extending* $p$ *which does not fork over* $M$*.*
4. *(Uniqueness) If* $M \leq_{\mathbf{K}} N$ *both in* $\mathbf{K}_{[\lambda,\mu)}$*,* $p, q \in \mathbf{S}^{bs}(N)$ *both do not fork over* $M$*, and* $p \upharpoonright M = q \upharpoonright M$*, then* $p = q$*.*
5. *(Strong Continuity[1]) If* $\delta < \mu$ *a limit ordinal,* $\langle M_i \mid i \leq \delta \rangle$ *increasing and continuous,* $\langle p_i \in \mathbf{S}^{bs}(M_i) \mid i < \delta \rangle$*, and* $i < j < \delta$ *implies* $p_j \upharpoonright M_i = p_i$*, and* $p_\delta \in \mathbf{S}(M_\delta)$ *is an upper bound for* $\langle p_i \mid i < \delta \rangle$*, then* $p_\delta \in \mathbf{S}^{bs}(M_\delta)$*. Moreover, if each* $p_i$ *does not fork over* $M_0$ *then neither does* $p_\delta$*.*

Now we introduce the notion of w*-good frames, which is related to w-good frames [MA20]. Although the author cannot find a proof or counterexample, w-good and w*-good frames are likely to be incomparable.

**Definition 129.** *A pre-*$[\lambda, \mu)$*-frame* $\mathfrak{s} = (\mathbf{K}, \downarrow, \mathbf{S}^{bs})$ *is a* w*-good *frame if* $\mathfrak{s}$ *satisfies:*

1. $\mathbf{K}_{[\lambda,\mu)}$ *has amalgamation, joint embedding and no maximal model.*
2. *(Uniqueness). See Definition 128.*
3. *(Basic NIP) For all* $M \in \mathbf{K}_{[\lambda,\mu)}$ $|\mathbf{S}^{bs}(M)| \leq ded\, \|M\|$*.*
4. *(Few Non-Basic Types) For all* $M \in \mathbf{K}_{[\lambda,\mu)}$*,* $|\mathbf{S}(M) - \mathbf{S}^{bs}(M)| \leq \lambda$*.*
5. *(Continuity[2]) Let* $\delta < \mu$ *be a limit ordinal,* $\langle M_i \mid i \leq \delta \rangle$ *increasing and continuous,*

[1] This was called just continuity in [MA20]. The author would like to thank Marcos Mazari-Armida for pointing out that the continuity axiom of a good frame requires only the moreover part.

[2] This is the continuity axiom for good frames.

*$\langle p_i \in \mathbf{S}^{bs}(M_i) \mid i < \delta\rangle$, and $i < j < \delta$ implies $p_j \restriction_{M_i} = p_i$, and $p_\delta \in \mathbf{S}(M_\delta)$ is an upper bound for $\langle p_i \mid i < \delta\rangle$. If each $p_i$ does not fork over $M_0$ then $p_\delta \in \mathbf{S}^{bs}(M_\delta)$ and $p_\delta$ also does not fork over $M_0$.*

6. *(Transitivity) if $p \in \mathbf{S}^{bs}(M_2)$ does not fork over $M_1 \leq_{\mathbf{K}} M_2$, and $p \restriction M_1$ does not fork over $M_0 \leq_{\mathbf{K}} M_1$, then $p$ does not fork over $M_0$.*

**Remark 130.** *(Continuity) is weaker than (Strong Continuity). Without not forking over $M_0$ one cannot deduce that $p_\delta \in \mathbf{S}^{bs}(M_\delta)$.*

**Remark 131.** *In a w-good frame (Transitivity) is implied by several other properties including (Existence of Non-Forking Extension). For a w*-good frame, where (Existence of Non-Forking Extension) does not hold in general, we need to explicitly include (Transitivity) as an axiom.*

**Definition 132.** *When $\mu = \lambda^+$ in the previous definitions, we say $\mathfrak{s}$ is a pre-/w-good/w*-good $\lambda$-frame.*

From now on we build a w*-good $\lambda$-frame on $K$ assuming the following:

**Hypothesis 133.** *($2^\lambda < 2^{\lambda^+}$) We fix $\mathbf{K}$ an AEC and a cardinal $\lambda \geq LS(\mathbf{K})$ such that $\mathbf{K}$ is categorical in $\lambda$. Furthermore $1 \leq \mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$, and $\mathbf{K}_\lambda$ has NIP.*

As $\mathbf{K}$ is categorical in $\lambda$, then $\mathbf{K}_\lambda$ has amalgamation by Fact 49. $\lambda$-JEP follows from categoricity, and $\lambda$-NMM follows from categoricity and $\mathbf{K}_{\lambda^+} \neq \emptyset$.

Recall the following definition:

**Definition 134.** *$p = \mathbf{gtp}(a/M, N)$ has the ($\lambda$-)extension property if for every $\mathbf{K}$-embedding $f : M \to M_1 \in \mathbf{K}_\lambda$ there is $q \in \mathbf{S}^{na}(M_1)$ extending $f(p)$.*

**Definition 135.** *$p = \mathbf{gtp}(a/M, N)$ is* $\lambda$-unique[3]. *if*

1. *$p = \mathbf{gtp}(a/M, N)$ has the extension property, and*
2. *for every $M \leq_{\mathbf{K}} M' \in \mathbf{K}_\lambda$, $p$ has at most one extension $q \in \mathbf{S}^{na}(M')$ with the extension property.*

**Fact 136.** *[She09b, §VI.2.5(2B)] If $\mathbf{K}_\lambda$ has the amalgamation property and $\lambda \geq LS(\mathbf{K})$, $\mathbf{gtp}(a, M, N)$ has $\geq \lambda^+$ realizations in some extension of $M$ (necessarily in $\mathbf{K}_{\geq \lambda^+}$) if and only if $\mathbf{gtp}(a/M, N)$ has the extension property.*

[3]This notion was first introduced by Shelah in [She75a, 6.1], called minimal types there. Note that this is a different notion from the minimal types of [She01]. These types are also called *quasiminimal types* in the literature, see for example [Zil05] and [Les05]

Now we define the w*-good $\lambda$-frame.

**Definition 137.** *The preframe $\mathfrak{s}_{\lambda-unq}$ is defined such that:*

1. $\mathbf{S}^{bs}(M) := \{p \in \mathbf{S}^{na}(M) \mid p$ *has the extension property*$\}$.
2. $p = \mathbf{gtp}(a/M, N) \in \mathbf{S}^{bs}(M)$ *does not fork over* $M_0 \leq_{\mathbf{K}} M$ *if* $p \upharpoonright M_0$ *is* $\lambda$*-unique.*

**Lemma 138.** $\mathbf{S}^{\lambda-al}(M)$[4] *satisfies* $|\mathbf{S}^{\lambda-al}(M)| \leq \lambda$*. By realizations we mean realizations in any* $\leq_{\mathbf{K}}$*-extension of* $M$ *in* $\mathbf{K}_{\lambda^+}$*. So* $\mathfrak{s}_{\lambda-unq}$ *satisfies (Few Non-Basic Types).*

*Proof.* Suppose not, i.e. $|\mathbf{S}^{\lambda-al}(M)| \geq \lambda^+$.

**Claim:** There is no $\Gamma \subseteq \mathbf{S}^{\lambda-al}(M)$, $|\Gamma| \leq \lambda$ that is inevitable.

Otherwise, suppose there exists such $\Gamma$. By Fact 44, taking $\mathbf{S}_*$ to be $\mathbf{S}^{na}$, and $\Gamma_M$ to be $\Gamma$, we have $|\mathbf{S}(M)| \leq \lambda$, so in particular $|\mathbf{S}^{\lambda-al}(M)| \leq \lambda$, contradiction.

Now by the claim and Fact 68, taking $\mathbf{S}_*$ there to be $\mathbf{S}^{\lambda-al}$ and $\mu$ there to be $\lambda^+$, we have $\mathbb{I}(\mathbf{K}, \lambda^+) = 2^{\lambda^+}$, contradiction. □

Thus from now on in this section we also assume $|\mathbf{S}^{\lambda-al}(M)| \leq \lambda$.

**Lemma 139.** $\mathfrak{s}_{\lambda-unq}$ *satisfies the following properties in Definitions 127, 128 and 129:*

1. *(Invariance);*
2. *(Monotonicity);*
3. *(Non-Forking Types are Basic);*
4. *amalmgamation, joint embedding and no maximal model;*
5. *(Basic NIP);*
6. *(Uniqueness);*
7. *(Transitivity).*

*Proof.* Easy. We prove (Transitivity) as an example. Suppose $p \in S^{bs}(N)$ does not fork over $M_1 \leq_{\mathbf{K}} N$, and $p \upharpoonright_{M_1}$ does not fork over $M_0 \leq_{\mathbf{K}} M_1$. Then $(p \upharpoonright_{M_1}) \upharpoonright_{M_0}$ is $\lambda$-unique, i.e. $p \upharpoonright_{M_0}$ is. Thus $p$ does not fork over $M_0$. □

The following property is essential for the next lemma.

**Definition 140.** *A type family* $\mathbf{S}_*$ *is* $\lambda$*-compact if for every limit ordinal* $\delta < \lambda^+$*, for every* $\langle M_i \in \mathbf{K}_\lambda : i < \delta\rangle$ *an increasing continuous chain and for every coherent sequence of types*

[4]Recall this is $\{p \in gS(M) \mid p$ has $\leq \lambda$-many realizations$\}$

$\langle p_i \in \mathbf{S}_*(M_i) : i < \delta \rangle$, *there is an upper bound* $p \in \mathbf{S}_*(\bigcup_{i<\delta} M_i)$ *to the sequence such that* $\langle p_i \in \mathbf{S}_*(M_i) : i < \delta + 1 \rangle$ *is a coherent sequence.*

For certain results in this chapter we need to assume that the basic types (i.e. those with the extension property) is $\lambda$-compact, which, for example, holds for abstract elementary classes with the disjoint amalgamation property, where every type has the extension property. Many classes of modules have the disjoint amalgamation property. See [MAR23, 2.10] and [BET07, 2.2]. Also, this assumption also holds in quasiminimal abstract elementary classes, where there is at most one non-algebraic type.

**Lemma 141** ($\lambda^+ < 2^\lambda$)**.** *Suppose that* $\mathbf{K}$ *is almost stable in* $\lambda$*, and* $\mathbf{S}^{bs}$ *of* $\mathfrak{s}_{\lambda-unq}$ *is* $\lambda$*-compact. If* $p \in \mathbf{S}^{bs}(M)$*, then there is* $N \geq_{\mathbf{K}} M$ *and* $q \in \mathbf{S}^{bs}(N)$ *extending* $p$ *that does not fork over* $N$*. In particular, for any* $N' \geq_{\mathbf{K}} N$ *there is unique* $q' \in \mathbf{S}(N')$ *extending* $q$ *that does not fork over* $N$*.*

*Proof.* It suffices to show that there is a $\lambda$-unique type above any basic type. By Fact 124 let $\mathfrak{C} \in \mathbf{K}_{\lambda^+}$ be saturated in $\lambda^+$ above $\lambda$. It is also homogeneous in $\lambda^+$ above $\lambda$ by Fact 30. Let $(a, M, N) \in \mathbf{K}^3_\lambda$ such that $\mathbf{gtp}(a/M, N)$ has the extension property and there is no $\lambda$-unique type above $\mathbf{gtp}(a/M, N)$. Build $(a_\eta, M_\eta, N_\eta) \in \mathbf{K}^3_\lambda$ for $\eta \in {}^{<\lambda}2$ and $h_{\eta,\nu}$ for $\eta < \nu \in {}^{<\lambda}2$ such that:

1. $(a_{\langle\rangle}, M_{\langle\rangle}, N_{\langle\rangle}) = (a, M, N)$.
2. $(a_\eta, M_\eta, N_\eta) \leq_{h_{\eta,\nu}} (a_\nu, M_\nu, N_\nu)$ for $\eta < \nu$.
3. $h_{\eta,\rho} = h_{\nu,\rho} \circ h_{\eta,\nu}$ for $\eta < \nu < \rho$.
4. $M_{\eta^\frown 0} = M_{\eta^\frown 1}$, $N_{\eta^\frown 0} = N_{\eta^\frown 1}$, and $h_{\eta,\eta^\frown 0} \restriction M_\eta = h_{\eta,\eta^\frown 1} \restriction M_\eta$.
5. $\mathbf{gtp}(a_{\eta^\frown 0}, M_{\eta^\frown 0}, N_{\eta^\frown 0}) \neq \mathbf{gtp}(a_{\eta^\frown 1}, M_{\eta^\frown 1}, N_{\eta^\frown 1})$, both having $\lambda^+$-many realizations.
6. If $\eta \in {}^\delta 2$ for $\delta$ a limit ordinal, take $M_\eta$ and $N_\eta$ to be directed colimits.

**Construction:** Base case and limit case are clear. At successor stage use non-$\lambda$-uniqueness to get two distinct extensions, each having $\lambda^+$-many realizations.

**Enough:** Let $M \leq_{\mathbf{K}} \mathfrak{C} \in \mathbf{K}_{\lambda^+}$ be saturated above $\lambda$. Build $g_\eta : M_\eta \to \mathfrak{C}$ for $\eta \in {}^{\leq\lambda}2$ such that:

1. $g_\nu \circ h_{\eta,\nu} = g_\eta$ for $\nu < \eta$.
2. $g_{\eta^\frown 0} = g_{\eta^\frown 1}$

**This is possible:** We carry out the construction by induction on the $\ell(\eta)$, the length of $\eta$. For the base case take $g_{\langle\rangle}$ to be inclusion $M \leq_{\mathbf{K}} \mathfrak{C}$. At limit use the universal property of $M_\eta$ as a directed colimit. For the successor case, for $\eta$ of length $\alpha = \beta + 1$, suppose we have $g_\eta$.

$$\begin{array}{ccccccc} \mathfrak{C} & \xleftarrow{\;j\;} & M''_{\eta^\frown 0} & \xleftarrow{\cong_h} & M'_{\eta^\frown 0} & \xrightarrow{\cong_g} & M_{\eta^\frown 0} \\ & {\nwarrow}^{id} & \uparrow id & & \uparrow id & & \uparrow id \\ & & g_\eta[M_\eta] & \xleftarrow{\cong_{g_\eta}} & M_\eta & \xrightarrow{\cong h_{\eta,\eta^\frown 0}} & h_{\eta,\eta^\frown 0}[M_\eta] \end{array} \tag{5.1}$$

Use basic extension to obtain the right square and $g$, and then obtain the middle square and $h$. Finally the left triangle is by saturation of $\mathfrak{C}$. Now define $g_{\eta^\frown 0} = g_{\eta^\frown 1}$ to be the composition of the top row from right to left.

**This is enough:** For each branch $\eta \in {}^{\lambda}2$, take directed colimit to obtain $(a_\eta, M_\eta, N_\eta)$. Obtain $f_\eta : M_\eta \to \mathfrak{C}$ by the universal property of colimits such that $f_\eta \circ h_{\nu,\eta} = g_\nu$ for all $\nu < \eta$, and obtain $f'_\eta : N_\eta \to \mathfrak{C}$ extending $f_\eta$ by saturation over $\lambda$. Since each $f'_\eta(a_\eta) \in |\mathfrak{C}|$, but $\|\mathfrak{C}\| = \operatorname{ded} \lambda < 2^\lambda$, there must be $\eta, \nu \in {}^{\lambda}2$ such that $f'_\eta(a_\eta) = f'_\nu(a_\nu)$. Let $\alpha < \lambda$ be the least such that $\eta(\alpha) \neq \nu(\alpha)$. Without loss of generality say $\eta(\alpha) = 0$ and $\nu(\alpha) = 1$. Then the following diagram commutes:

$$\begin{array}{ccc} N_{(\eta\restriction\alpha)^\frown 0} & \xrightarrow{f'_\eta \circ h_{(\eta\restriction\alpha)^\frown 0,\eta}} & \mathfrak{C} \\ \uparrow id & f'_\nu \circ h_{(\eta\restriction\alpha)^\frown 1,\nu} & \uparrow \\ M_{(\eta\restriction\alpha)^\frown 0} & \xrightarrow{id} & N_{(\eta\restriction\alpha)^\frown 1} \end{array} \tag{5.2}$$

with $f'_\eta \circ h_{(\eta\restriction\alpha)^\frown 0,\eta}(a_{(\eta\restriction\alpha)^\frown 0}) = f'_\nu \circ h_{(\eta\restriction\alpha)^\frown 1,\nu}(a_{(\eta\restriction\alpha)^\frown 1})$ since $f'_\eta(a_\eta) = f'_\nu(a_\nu)$, contradicting requirement (5) of the construction. □

**Remark 142.** *The proof of Lemma 141 is along the argument of Mazari-Armida in [MA20, 4.13] and [She09b, VI.2.25], and the difference is that there the saturated model over $\lambda$ lies in $K_{\lambda^{++}}$. For completeness we included all the details.*

**Question 143.** *Lemma 141 is a weaker form of (Existence of Non-Forking Extension). Is it possible to obtain (Existence of Non-Forking Extension) in its full strength, by perhaps considering another family of basic types and non-forking relation? One could imitate the w-good $\lambda$-frame in [MA20] and use $\lambda$-unique types as basic ones, and then Lemma 141 gives a proof of (Weak Density). However, then it is hard to show that having such a frame implies NIP.*

Recall the definition of locality:

**Definition 144.** *1.* $\mathbf{K}$ *is* $(\kappa, \lambda)$-local *if for every increasing continuous chain* $M = \bigcup_{i<\kappa} M_i$ *with* $\|M\| = \lambda$ *and for any* $p, q \in gS(M)$*: if* $p \restriction M_i = q \restriction M_i$ *for all* $i$ *then* $p = q$.

*2.* $K$ *is* $(< \kappa, \lambda)$-local *if* $K$ *is* $(\mu, \lambda)$*-local for all* $\mu < \kappa$.

**Lemma 145.** *If* $\mathbf{K}$ *is* $(< \lambda^+, \lambda)$*-local, then* $\mathfrak{s}_{\lambda-unq}$ *has (Continuity).*

*Proof.* Let $\langle M_i : i < \delta \rangle$ be increasing continuous. $p_i \in \mathbf{S}^{bs}(M_i)$ increasing and for $i < j < \delta$ we have $p_j \restriction M_i = p_i$, all non-forking over $M_0$ and $p_\delta$ upper bound. Suppose $p_\delta$ has $\leq \lambda$-many realizations. Then there is a set $S$ of cardinality $\lambda^+$ of realizations of $p_0$, such that for each $a \in S$, by locality there is $i < \delta$ such that $a$ realizes $p_i$ but not $p_{i+1}$, as the type of that realization over $\bigcup_i M_i$ is not $p_\delta$. By pigeonhole principle for some $i < \delta$ there are $\lambda^+$-many realizations of $p_i$ that are not realizations of $p_{i+1}$. Since there are $\leq \lambda$-many types in $\mathbf{S}(M_{i+1})$ that have $\leq \lambda$-many realizations, there must be another type in $\mathbf{S}(M_{i+1})$ with $\lambda^+$ realizations distinct from $p_{i+1}$, which contradicts $\lambda$-uniqueness of $p_i$.

For the moreover part, if $p_0$ does not fork over $M_0$, so $p_0 = p_\delta \restriction M_0$ is $\lambda$-unique, i.e. $p_\delta$ does not fork over $M_0$. □

**Theorem 146.** *(*$2^\lambda < 2^{\lambda^+}$*) Let* $\mathbf{K}$ *be an abstract elementary class categorical in* $\lambda \geq LS(\mathbf{K})$*, and* $1 \leq \mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$*.* $\mathbf{K}_\lambda$ *has NIP if and only if there is a w*-good* $\lambda$*-frame on* $\mathbf{K}$ *except possibly without (Continuity). Moreover,*

1. *(*$\lambda^+ < 2^\lambda$*) If* $\mathbf{K}$ *is almost stable in* $\lambda$ *and* $\mathfrak{s}_{\lambda-unq}$ *is* $\lambda$*-compact, then the w*-good frame satisfies in addition that if* $p \in \mathbf{S}^{bs}(M)$*, then there is* $N \geq_{\mathbf{K}} M$ *and* $q \in \mathbf{S}^{bs}(N)$ *extending* $p$ *that does not fork over* $N$*. In particular, for any* $N' \geq_{\mathbf{K}} N$ *there is* $q' \in \mathbf{S}(N')$ *extending* $q$ *that does not fork over* $N$*.*
2. *Almost stability in the previous part can be replaced with NIP in* $\lambda^+$*.*
3. *If* $\mathbf{K}$ *is* $(< \lambda^+, \lambda)$*-local, then* $\mathfrak{s}_{\lambda-unq}$ *has (Continuity).*

*Proof.* It suffices to prove the second part as the other parts are already proved. We use [She09b, VI.4.2]. Note that clause $(\eta)$ there implies ded $(\lambda^+) = 2^{\lambda^+}$, so case $(A)_\lambda$ there holds. We then would like to use [She09b, VI.4.5]. Although we do not have clause (e) there, it is only used to allow constructions done in [She09b, 2.3], which can also be done assuming the failure of the conclusion. □

## 5.3 Syntactic independence property

In this section we assume tameness, and use Galois Morleyization to show that the negation of NIP leads to being able to encode subsets, as a parallel of first order independence property.

**Hypothesis 147.** *Let $\kappa$ be an infinite cardinal and $\mathbf{K}$ an AEC. Let $\tau = L(K)$ be its underlying language.*

The following technique first appeared in [Vas16c], which allows one to work with Galois types in a syntactic way.

**Definition 148.** *Let $\kappa$ be an infinite cardinal and $\mathbf{K}$ an abstract elementary class. The* $(<\kappa)$-Galois Morleyization *of $\mathbf{K}$ is $\hat{\mathbf{K}}$, an AEC (except that the language might not be finitary) in a $(<\kappa)$-ary language $\hat{\tau}$ extending $\tau$ such that:*

1. *The structures and the substructure relation $\leq_{\hat{\mathbf{K}}}$ in $\hat{\mathbf{K}}$ are the same as $\mathbf{K}$.*
2. *For each $p \in \mathbf{S}^{<\kappa}(\emptyset)$, there is a predicate of the same length $R_p \in \hat{\tau}$. For each $M \in \mathbf{K}$ and $\bar{a} \in |M|$, define $M \models R_p[\bar{a}]$ if and only if $\mathbf{gtp}(\bar{a}/\emptyset, M) = p$. By extension, one can interpret quantifier-free $\mathbb{L}_{\kappa,\kappa}(\hat{\tau})$ formulas.*
3. *The $(<\kappa)$-syntactic type of $\bar{a} \in {}^{<\kappa}|M|$ over $A \subseteq |M|$ is $\boldsymbol{tp}_{qf\text{-}\mathbb{L}_{\kappa,\kappa}(\hat{\tau})}(\bar{a}/A, M)$, the set of all quantifier-free $\mathbb{L}_{\kappa,\kappa}(\hat{\tau})$ formulas with parameters from $A$ that $\bar{a}$ satisfies. For a particular quantifier-free $\mathbb{L}_{\kappa,\kappa}(\hat{\tau})$-formula $\phi(\bar{x}, \bar{y})$,*
   *$\boldsymbol{tp}_{\phi}(\bar{b}/A, M) := \{\phi(\bar{x}, \bar{a}) \mid \bar{a} \in A, M \models \phi(\bar{b}, \bar{a})\}$.*
4. *For $M \in \mathbf{K}$ and $A \subseteq |M|$, $\mathbf{S}^{<\alpha}_{qf-\mathbb{L}_{\kappa,\kappa}(\hat{\tau})}(A; M) := \{\boldsymbol{tp}_{qf\text{-}\mathbb{L}_{\kappa,\kappa}(\hat{\tau})}(\bar{b}/A, M) \mid \bar{b} \in {}^{<\alpha}|M|\}$*

**Remark 149.** *There are $\leq 2^{<(LS(\mathbf{K})^{+}+\kappa)}$ formulas in $\hat{\tau}$.*

**Definition 150.** *For a theory $T$ in first order logic, and $\Gamma$ a set of $T$-types, $\tau$ a language contained in the language of $T$, let $EC(T, \Gamma)$ denote the class of models of $T$ omitting all types in $\Gamma$. Let $PC(T, \Gamma, \tau)$ denote the class of models of $T$ omitting all types in $\Gamma$ as $\tau$-structures.*

**Fact 151.** *[Vas16c, 3.18(2)] Under the notation of the previous definition, $\mathbf{K}$ is $(<\kappa)$-tame if and only if for each ordinal $\alpha$, $M \in \mathbf{K}$, $A \subseteq M$, $\mathbf{gtp}(\bar{b}/A, M) \mapsto \boldsymbol{tp}_{qf\text{-}\mathbb{L}_{\kappa,\kappa}(\hat{\tau})}(\bar{b}/A, M)$ from $\mathbf{S}^{\alpha}(A; M)$ to $\mathbf{S}^{\alpha}_{qf-L_{\kappa,\kappa}(\hat{\tau})}(A; M)$ is bijective.*

**Definition 152.** *For $T$ a first order theory, $\Gamma$ a set of $T$-types, let $EC(T, \Gamma)$ denote the class of $T$-models that omit all types in $\Gamma$. If moreover $\tau$ is a language such that all of its* *symbols appear in the language of $T$, let $PC(T,\Gamma,\tau)$ denote the class of $T$ models omitting each type in $\Gamma$ interpreted as $\tau$-structures.*

**Theorem 153.** *Suppose $\mathbf{K}$ is $(<\aleph_0)$-tame, $M \in \mathbf{K}$, $C \subseteq |M|$, $\lambda := |C| \geq \beth_3(LS(\mathbf{K}))$ and $(\text{ded}\ \lambda)^{2^{LS(\mathbf{K})}} = \text{ded}\ \lambda$. Suppose $|\mathbf{S}^1(C;M)| > \text{ded}\ \lambda$. Then there is $N \in \mathbf{K}$, $\langle \bar{a}_n \in {}^m|N| \mid n < \omega\rangle$ and $\phi$ in the language of the Galois Morleyization of $\mathbf{K}$ such that for every $w \subseteq \omega$ there is $b_w \in |N|$ such that for all $i < \omega$,*

$$N \models \phi(\bar{a}_i, b_w) \iff i \in w.$$

*Proof.* Let $\hat{\mathbf{K}}$ be the $(<\aleph_0)$ Galois Morleyization of $\mathbf{K}$. Note that both classes have the same Galois types. By Shelah's Presentation Theorem $\hat{\mathbf{K}} = PC(T,\Gamma,\hat{\tau})$ with $|T| \leq 2^{LS(\mathbf{K})}$, with the language of $T$ containing $\hat{\tau}$. Then by tameness and the previous fact $|\mathbf{S}^1_{\text{qf}-\mathbb{L}_{\omega,\omega}(\hat{\tau})}(C;M)| > \text{ded}\ \lambda$, so for some quantifier-free formula $\phi(\bar{y},x)$ in $\mathbb{L}_{\omega,\omega}(\hat{\tau})$ with $|\mathbf{S}_\phi(C;M)| > \text{ded}\ \lambda$, since there are $\leq 2^{LS(\mathbf{K})}$-many quantifier-free $\mathbb{L}_{\omega,\omega}(\hat{\tau})$-formulas.

Without loss of generality $C = \lambda = |C|$. Let $\mu := (\text{ded}\ \lambda)^+$. For notational simplicity we view $\mathbf{S}_\phi(C;M)$ as $S$, a family of subsets of ${}^{\ell(\bar{y})}C$, where

$$A \in S \iff \{\phi(\bar{a},x) \mid \bar{a} \in A\} \in \mathbf{S}_\phi(C;M).$$

We also assume $\bar{y}$ has length 1. The proof for other cases is similar.

**Claim:** For all $\alpha < \lambda$, if $|\{A \cap \alpha \mid A \in S\}| \geq \mu$, then $\alpha \geq (\beth_2(LS(\mathbf{K})))^+$.

*Proof of Claim:* Suppose there is $\alpha < \lambda$, $|\{A \cap \alpha \mid A \in S\}| \geq \mu$. Since $\{A \cap \alpha \mid A \in S\}$ is the set of branches of the a subtree of ${}^{<\alpha}2$, $\text{ded}\ \lambda < \mu \leq \text{ded}\ |{}^{<\alpha}2| \leq \text{ded}\ 2^{|\alpha|}$, so $2^{|\alpha|} > \lambda \geq \beth_3(LS(\mathbf{K}))$, so $|\alpha| > \beth_2(LS(\mathbf{K}))$. Thus the claim holds.

We may assume $\lambda > \beth_2(LS(\mathbf{K}))$ and for all $\alpha < \lambda$, $|\{A \cap \alpha \mid A \in S\}| < \mu$. If this holds, then we are done since $\lambda \geq \beth_3(LS(\mathbf{K})) > \beth_2(LS(\mathbf{K}))$. If not, replace $\lambda$ with smallest $\alpha < \lambda$ such that $|\{A \cap \alpha \mid A \in S\}| \geq \mu$. By minimality for all $\beta < \alpha$, $|\{A \cap \beta \mid A \in S\}| < \mu$. Such $\alpha$ might be small, but by the claim $\alpha \geq (\beth_2(LS(\mathbf{K})))^+$, and this is enough for the rest of the argument.

For each $\alpha \leq \lambda$ let $S^0_\alpha := \{\langle A \cap \alpha, \alpha\rangle \mid A \in S\}$. $\bigcup_{\alpha<\lambda} S^0_\alpha$ is a tree when equipped with

$$(A_1,\alpha_1) \leq (A_2,\alpha_2) \iff \alpha_1 \leq \alpha_2 \wedge A_1 = A_2 \cap \alpha_1.$$

Let

$$S^1_\alpha := \{s \in S^0_\alpha \mid |\{t \in S^0_\alpha \mid s \leq t\}| \geq \mu\},$$

and

$$S^1_\lambda := \{s \in S^0_\lambda \mid \forall \alpha < \lambda (s \restriction_\alpha \in S^1_\alpha)\}.$$

We build

1. for $n < \omega$, $S_n \subseteq S^1_\lambda$, and
2. for each $(A, \lambda) \in S_n$ and $i < \lambda$,
   (a) $\lambda > \alpha_i^{A \cap i}(n, 0) > \ldots > \alpha_i^{A \cap i}(n, n-1) > i$, a sequence of ordinals,
   (b) $(D_{u,n}^{(A\cap i, i)}, \lambda) \in S^1_\lambda$ for each $u \subseteq n$, and
3. $p_n \in S_T^{n+2^n}(\emptyset)$ for $n < \omega$

such that:

1. $S_0 = S^1_\lambda$;
2. $|S_n| \geq \mu \geq (\beth_2(LS(\mathbf{K})))^+$ for all $n$;
3. $S_{n+1} \subseteq S_n$ for all $n$;
4. The variables of $p_n$ are $x_i$ for $i < n$ ordered naturally and $y_S$ for $S \subseteq n$;
5. $p_n \subseteq p_{n+1}$ for all $n$. This means the $p_{n+1}$ restricted to $x_i$ for $i < n$ and $y_S$ for $S \subseteq n$ is equal to $p_n$;
6. For all $n < m$, $(A, \lambda) \in S_n$ and $(B, \lambda) \in S_m$, $i, j \in \lambda$

$$\begin{aligned} p_n &= \mathbf{tp}_T(\langle \alpha_i^{A\cap i}(n, 0), \ldots \alpha_i^{A\cap i}(n, n-1)\rangle^\frown \langle D_{w,n}^{(A\cap i, i)} \mid w \subseteq n\rangle/\emptyset, M) \\ &= \mathbf{tp}_T(\langle \alpha_j^{B\cap j}(m, 0), \ldots \alpha_j^{B\cap j}(m, n-1)\rangle^\frown \langle D_{w,m}^{(B\cap j, j)} \mid w \subseteq m\rangle/\emptyset, M); \end{aligned}$$

7. For all $(A, i) \in S_n$ and $w \subseteq n$, $(A, i) \leq (D_{w,n}^{(A,i)}, \lambda)$ and $\alpha_i^A(n, i) \in D_{w,n}^{(A,i)} \iff i \in w$.

**Construction:** We build these objects by induction on $n$. When $n = 0$ let $D_{\emptyset,0}^{(\emptyset,0)}$ be any element in $S^1_\lambda$. Assume we have built $S_n$, $\alpha_i^{A\cap i}(n, j)$ for $(A, \lambda) \in S_n$ and $p_n$.

Fix $s = (A, i)$ for $i < \lambda$ such that for some $B$, $A \subseteq B$ and $(B, \lambda) \in S_n$. Clearly $T_s := \{t \in \bigcup_{\beta<\lambda} S^1_\beta \mid s \leq t$ and $t$ extends to an element in $S_n\}$ is a tree. For every $s \leq t \in S_n$, $B_t := \{t^* \mid s \leq t^* \leq t\}$ is a branch of $T_s$, and $t_1 \neq t_2 \implies B_{t_1} \neq B_{t_2}$. Since

$$|S^0_\lambda - S^1_\lambda| = |\bigcup_{\alpha<\lambda, s \in S^0_\alpha - S^1_\alpha} \{t \in S^0_\lambda \mid s \leq t\}| < \mu,$$

$T_s$ has $\geq \mu$-many branches, and hence $|T_s| > \lambda$. Then for some $i'$, $|T_s \cap S^1_{i'}| > \lambda$. Let $s_j = (A_j, i') \in T_s \cap S^1_{i'}$ for $j < \lambda^+$. Since there are $\leq \lambda$ finite tuples of ordinals $< \lambda$, we may assume $\alpha_{i'}^{A_j}$ are the same for all $j$. Now let $\alpha_i^A(n+1,k) := \alpha_{i'}^{A_j}(n,k)$ for all $k < n$. Let $\alpha_i^A(n+1,n)$ be the least $\alpha$ such that $s_0(\alpha) \neq s_1(\alpha)$, i.e. $\alpha \in A_0 - A_1$ or $\alpha \in A_1 - A_0$. Without loss of generality assume the latter case. For $w \subseteq (n+1)$, let $D_{w,n+1}^{(A,i)} := D_{w,n}^{(A_0,i')}$ if $n \notin w$ and $D_{w,n+1}^{(A,i)} := D_{w,n}^{(A_1,i')}$ if $n \in w$.

Note that $i < \alpha_i^A(n+1,n) < i' < \alpha_i^A(n+1,n-1) < \ldots < \alpha_i^A(n+1,0)$. Since $|S_n| \geq (\beth_2(LS(\mathbf{K})))^+$, and there are $\leq \beth_2(LS(\mathbf{K}))$ $T$-types, by the pigeonhole principle there is $S_{n+1} \subseteq S_n$, $|S_{n+1}| \geq (\beth_2(LS(\mathbf{K})))^+$ such that for all $(A,i)$, $(B,j) \in S_{n+1}$,

$$\mathbf{tp}_T(\langle \alpha_i^A(n,0), \ldots \alpha_i^A(n+1,n)\rangle^\frown \langle D_{w,n+1}^{(A,i)} \mid w \subseteq n+1\rangle/\emptyset, M)$$

is the same, and define this type to be $p_{n+1}$. This finishes the construction. Note that here since $D_{w,n+1}^{(A,i)}$ is an element of $S_\lambda^1 \subseteq S_\lambda^0 = S$, i.e. a $\phi$-type, the "$T$-type" of $D_{w,n+1}^{(A,i)}$ is just the $T$-type of a realization of it, which can be fixed at the beginning of the proof.

$$T^* := T \cup \{\phi(c_i, d_w)^{i \in w}) \mid w \subseteq \omega\} \cup \{p_n(\langle c_i \mid i < n\rangle^\frown \langle d_w \mid w \subseteq \omega\rangle) \mid n < \omega\}$$

is consistent, and by Morley's method we are done. □

Similar to the order property, this analogue of the independence property for AECs also has a Hanf number $\beth_{(2^{LS(\mathbf{K})})^+}$.

**Theorem 154.** *If $\mathbf{K}$ can encode subsets of $\mu := \beth_{(2^{LS(\mathbf{K})})^+}$, then it can encode subsets of any cardinal. That is, if there are $M \in \mathbf{K}$, $\{a_i \mid i < \mu\} \subseteq |M|$, $\{b_w \mid w \subseteq \mu\} \subseteq |M|$ such that for all $w \subseteq \mu$,*

$$i \in w \iff \phi(a_i, b_w),$$

*then we can replace $\mu$ above by any cardinal.*

*Proof.* We fix $\hat{\mathbf{K}}$ and $\phi$ as in the proof of the previous theorem. Let $\lambda = (2^{LS(\mathbf{K})})^+$. Suppose $\mathbf{K}$ can encode subsets of $\mu := \beth_{(2^{LS(\mathbf{K})})^+}$. That is, there are $M \in \mathbf{K}$, $\{a_i \mid i < \mu\} \subseteq |M|$, $\{b_w \mid w \subseteq \mu\} \subseteq |M|$ such that for all $w \subseteq \mu$,

$$i \in w \iff \phi(a_i, b_w).$$

For each $i_0 < \ldots < i_{n-1} < \mu$ and $u \subseteq n$, choose some subset $w \subseteq \mu$ such that $i_j \in w \iff \phi(a_{i_j}, b_w) \iff j \in u$, and let $b_{u,n}^{i_0,\ldots,i_{n-1}} := b_w$. We build $\langle F_n \subseteq \mu \mid n < \omega\rangle$, $\langle X_{\xi,n} \subseteq \mu \mid \xi \in F_n, n < \omega\rangle$ and $p_n \in \mathbf{S}_T^{n+2^n}(\emptyset)$ such that:

1. For all $n < \omega$, $|F_n| = \lambda$;
2. $|X_{\xi,n}| > \beth_\beta(2^{LS(\mathbf{K})})$ when $\xi$ is the $\beta$-th element of $F_n$;
3. $p_n(\langle a_{i_j} \mid j < n\rangle^\frown\langle b^{i_0,\dots,i_{n-1}}_{u,n} \mid u \subseteq n\rangle)$.

Let $F_0 = \lambda$ and $X_{\xi,0} := \mu$ for all $\xi$. Suppose we have constructed everything for stage $n$. Fix $g : \lambda \to F_n$ an increasing enumeration. Let $G_n := \{g(\beta + n + 1) \mid \beta < \lambda\}$. For each $\xi = g(\beta + n + 1) \in G_n$, consider the map $\langle i_j \mid j < n\rangle \mapsto \mathbf{tp}_T(\langle a_{i_j} \mid j < n + 1\rangle^\frown\langle b^{i_0,\dots,i_n}_{u,n+1} \mid u \subseteq n + 1\rangle/\emptyset, M)$ from $[X_{\xi,n}]^{n+1}$ (increasing $(n + 1)$-tuples from $X_{\xi,n}$) to $S^{n+2^n}_T(\emptyset)$. Since $|X_{\xi,n}| > \beth_{\beta+n+1}((2^{LS(\mathbf{K})})^+)$, by the Erdős-Rado theorem, there is a monochromatic subset $X_{\xi,n+1} \subseteq X_{\xi,n}$ such that $|X_{\xi,n+1}| > \beth_\beta((2^{LS(\mathbf{K})})^+)$. I.e. there is a type $p_{\xi,n+1}$ such that for all $i_0 < \ldots < i_n$, $\mathbf{tp}_T(\langle a_{i_j} \mid j < n\rangle^\frown\langle b^{i_0,\dots,i_n}_{u,n+1} \mid u \subseteq n + 1\rangle/\emptyset, M) = p_{\xi,n+1}$. By the pigeonhole principle there is $F_{n+1} \subseteq G_n$ of cardinality $\lambda$ and $p_{n+1}$ such that for all $\xi \in F_{n+1}$, $p_{\xi,n+1} = p_{n+1}$.

Then

$$T^* := T \cup \{\phi(c_i, d_w)^{i\in w}) \mid w \subseteq \kappa\} \cup \{p_n(\langle c_{i_j} \mid j < n\rangle^\frown\langle d_w \mid w \subseteq w\rangle) \mid n < \omega, i_0 < \ldots < i_{n-1} < \kappa\}$$

is consistent for any cardinal $\kappa$. By Morley's method we are done. □

The following well-known fact is usually called Morley's method. Usually it is stated in a slightly weaker way. We include a proof for the sake of completeness.

**Fact 155** (Morley's method)**.** *Let $T$ be a first order theory with built-in Skolem functions and $\Gamma$ a set of $T$-types. Let $\langle c_i \mid i < \alpha\rangle$ be new constants. Let $p_S$ be a $T$-type in $|S|$ variables for every finite subset $S$ of $\alpha$, and $T^*$ a theory not containing any of the new constants such that:*

1. *$T^* \supseteq T \cup \{p_S(\langle c_\gamma \mid \gamma \in S\rangle) \mid S \subseteq \alpha$ finite$\}$ is consistent;*
2. *Each $p_S$ is realized in some $M \in EC(T, \Gamma)$.*

*Then there is $N \in EC(T^*, \Gamma)$.*

*Proof.* Let $M$ be a model of $T^*$ and without loss of generality $M = EM(\{c_i \mid i < \alpha\})$. We show that $M$ omits all types from $\Gamma$. Suppose not, i.e. $a \in |M|$ realizes some $q \in \Gamma$. Write $a$ as $\tau^M(c_{i_0}, \dots, c_{i_k})$ for some term $\tau$ in the language of $T$. Let $S := \{c^M_{i_0}, \dots, c^M_{i_k}\}$ and $\langle b_0, \dots, b_k\rangle \subseteq N^* \in EC(T, \Gamma)$ realizing $p_S$. Then for some

$\varphi(y) \in q$, $N^* \models \neg\varphi(\tau(b_0, \ldots, b_k))$. As $p_S$ is complete, $\neg\varphi(\tau(x_0, \ldots, x_k)) \in p_S$. Thus $M \not\models \varphi(\tau(c_{i_0}, \ldots, c_{i_k}))$, i.e. $M \models \neg\varphi(a)$, so $a$ does not realize $q$. $\square$

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