AN NIP-LIKE NOTION IN ABSTRACT ELEMENTARY CLASSES

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Abstract. This paper is a contribution to the "neo-stability" type of result for abstract elementary classes. Under certain set theoretic assumptions, we propose a definition and a characterization of NIP in AECs. The class of AECs with NIP properly contains the class of stable AECs[1]. We show that for an AEC K and $\lambda \geq LS(K)$, K_λ is NIP if and only if there is a notion of nonforking on it which we call a w*-good frame. On the other hand, the negation of NIP leads to being able to encode subsets.

Contents

1. Introduction

There is a massive body of literature on "neostability" for first order theories dedicated to exploration and study of forking-like relations for various classes of unstable theories. The main classes: NIP theories, simple theories, theories with the strict order property, theories with the tree property of type 1 and 2, were all presented by Shelah in [She78]. In mid 1976 Shelah set the program which he named **classification theory for non-elementary classes**. A few years later the focus shifted to abstract elementary classes (AECs).

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[1]See Examples 2.20 and 2.21 for AECs that are unstable, not elementary but NIP.

An appropriate generalization of stability for AECs was introduced in [She99] building on many previous papers including [She71b] and [GS]. In the last forty years starting with [GS86] much was discovered about analogues of superstability. See [Vas16b], [GV17], and [Leu23] for some recent work.

In this paper we propose progress towards "neostability of AECs", more precisely, exploring an analogue of NIP and its negation. We propose a definition (under a certain cardinal arithmetic axiom) of NIP. Using techniques from papers by Shelah [She09a], Jarden and Shelah [JS13] and Mazari-Armida [MA20], we obtain a characterization of NIP in AECs using frames (a forking-like relation).

The notion of the λ-stable AEC was first studied in [She99] using non-splitting. Various frameworks of forking-like relations were introduced. In [She09a], Shelah introduced the local notion of the good λ-frame, an axiomatization of forking-like relations for structures of cardinality λ in AECs, as a parallel of superstability. In [BG17] Boney and Grossberg established that for "nice" AECs, stablity implies existence of strong independence relations on the subclass of saturated models, which allows types of arbitrary length. In [BGKV16] it was shown that this relation and several others are unique/canonical (if they exist).

Although good λ-frames are nice and powerful, sometimes they might not exist. There are several weaker notions, where some of the axioms of a good λ-frame are weakened or dropped. Vasey worked with good$^-$ λ-frames in [Vas16b] and good^{-S} λ-frames in [Vas16a]. Jarden and Shelah defined semi-good λ-frames in [JS13]. Mazari-Armida introduced w-good λ-frames in [MA20], which is weaker than all the axiomatic frames mentioned above.

Definition 1.1. Let K be an AEC, $\lambda \geq LS(K)$. K_λ has NIP if for all $M \in K_\lambda$, $|gS(M)| \leq \text{ded } \lambda$.

Our definition of NIP will be discussed further in the next section.

Our main results are:

Theorem 1.2 ($2^{\lambda^+} > 2^\lambda$)**.** Let K be an AEC categorical in $\lambda \geq LS(K)$, and $1 \leq I(\lambda^+, K) < 2^{\lambda^+}$. K_λ has NIP if and only if there is a w*-good λ-frame on K except possibly without (Continuity). Moreover,

(1) (ded $\lambda = \lambda^+ < 2^\lambda$) If $\mathfrak{s}_{\lambda-unq}$ is λ-compact, then the w*-good frame satisfies in addition that if $p \in S^{bs}(M)$, then there is $N \geq_K M$ and $q \in S^{bs}(N)$ extending p that does not fork over N. In particular, for any $N' \geq_K N$ there is $q' \in gS(N')$ extending q that does not fork over N.
(2) if K is $(< \lambda^+, \lambda)$-local, then $\mathfrak{s}_{\lambda-unq}$ has (Continuity).

Theorem 1.3. Suppose K is $(< \aleph_0)$-tame, $M \in K$, $C \subseteq |M|$, $\lambda := |C| \geq \beth_3(LS(K))$ and $(\text{ded } \lambda)^{2^{LS(K)}} = \text{ded } \lambda$. Suppose $|gS^1(C; M)| > \text{ded } \lambda$. Then there is $N \in K$, $\langle \bar{a}_n \in^m |N| \mid n < \omega \rangle$ and ϕ in the language of Galois Morleyization such that for every $w \subseteq \omega$ there is $b_w \in |N|$ such that for all $i < \omega$,

$$N \models \phi(\bar{a}_i, b_w) \iff i \in w$$

Theorem 1.4. If K can encode subsets of $\mu := \beth_{(2^{LS(K)})^+}$, then it can encode subsets of any cardinal. That is, if there are $M \in K$, $\{a_i \mid i < \mu\} \subseteq |M|$, $\{b_w \mid w \subseteq \mu\} \subseteq |M|$ such that for all $w \subseteq \mu$,

$$i \in w \iff \phi(a_i, b_w),$$

then we can replace μ above by any cardinal.

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It is interesting to comment that Shelah already implicitly discussed similar results in [She01] dealing with Grossberg's question "Does $I(\lambda, K) = I(\lambda^{++}, K) = 1$ imply $K_{\lambda^{++}} \neq \emptyset$" and in its followup [She09a], Chapter II of [She09c], and [She09b], Chapter VI of [She09d]. More specifically, in [She09d, VI.2.3] and [She09d, VI.2.5] Shelah considered the number of branches of a tree as a bound of Galois types over a model.

2. Preliminaries

Notation 2.1.

(1) For any structure M in some language, we denote its universe by $|M|$, and its cardinality by $\|M\|$.
(2) For cardinals λ and μ, $[\lambda, \mu) := \{\kappa \in \text{Card} \mid \lambda \leq \kappa < \mu\}$. $[\lambda, \infty) := \{\kappa \in \text{Card} \mid \lambda \leq \kappa\}$.
(3) $K_{[\lambda,\mu)} := \{M \in K \mid \|M\| \in [\lambda, \mu)\}$. $K_\lambda := K_{[\lambda,\lambda^+)}$

Definition 2.2. For K an AEC, we say:

(1) K has the amalgamation property (AP) if for all $M_0 \leq M_\ell$ for $\ell = 1, 2$, there is $N \in K$ and K-embeddings $f_\ell : M_\ell \to N$ for $\ell = 1, 2$ such that $f_1 \restriction_{M_0} = f_2 \restriction_{M_0}$.
(2) K has the joint embedding property (JEP) if for all M_0, $M_1 \in K$ there are $N \in K$ and K-embeddings $f_\ell : M_l \to N$ for $\ell = 0, 1$.
(3) K has no maximal models (NMM) if for all $M \in K$ there is $N >_K M$.

Remark 2.3. For a property P, e.g. amalgamation, we say that K_λ has P or that K has λ-P if we restrict to K_λ in the above definition.

Definition 2.4.

(1) $K^3_\lambda := \{(a, M, N) \mid M, N \in K_\lambda, M <_K N, a \in |N| - |M|\}$.
(2) For (a_0, M_0, N_0), $(a_1, M_1, N_1) \in K^3_\lambda$, we say $(a_0, M_0, N_0) \leq (a_1, M_1, N_1)$ if $M_0 \leq M_1$, $a_0 = a_1$ and $N_0 \leq_K N_1$.
(3) For (a_0, M_0, N_0), $(a_1, M_1, N_1) \in K^3_\lambda$ and K-embedding $h : N_0 \to N_1$, $(a_0, M_0, N_0) \leq_h (a_1, M_1, N_1)$ if $h \restriction_{M_0}: M_0 \to M_1$ is a K-embedding and $h(a_0) = a_1$.

Definition 2.5.

(1) For $(a_0, M_0, N_0), (a_1, M_1, N_1) \in K^3_\lambda$, $(a_0, M_0, N_0)E_{at}(a_1, M_1, N_1)$ if $M_0 = M_1$, and there are $N \in K$, $f_0 : N_0 \to N$, and $f_1 : N_1 \to N$ K-embeddings such that $f_0(a_0) = f_1(a_1)$ and $f_0 \restriction_{M_0} = f_1 \restriction_{M_0}$.
(2) E is the transitive closure of E_{at}.
(3) For $(a, M, N) \in K^3_\lambda$, *the Galois type of a over M in N* is $\mathbf{gtp}(a/M, N) := [(a, M, N)]_E$.
(4) For $M \in K_\lambda$, $gS(M) := \{\mathbf{gtp}(a/M, N) \mid (a, M, N) \in K^3_\lambda\}$.

For $M_0 \leq_K M \in K_\lambda$ and $p = \mathbf{gtp}(a/M, N) \in gS(M)$, define $p \restriction_{M_0} := \mathbf{gtp}(a/M_0, N)$.

For $M_0 \leq_K M_1$ and types $p \in gS(M_0)$ and $q \in gS(M_1)$, we say $p \leq q$ if $p = q \restriction_{M_0}$.

Remark 2.6. If K_λ has AP then $E_{at} = E$.

Definition 2.7. Assume that K_λ has AP. For M, $N \in K$, $p \in gS(M)$ and K-embedding $h : M \to N$, we define $h(p) := \mathbf{gtp}(h'(a)/h[M], N)$, where $h' : M' \to N'$ extends h and $(a, M, M') \in p$. Note that $h(p)$ does not depend on the choice of (a, M, M') or h'. See [Leu23, 3.1] for a proof.

Definition 2.8. Let $\langle M_i \mid i < \delta \rangle$ be increasing continuous. A sequence of types $\langle p_i \in gS(M_i) \mid i < \delta \rangle$ is coherent if there are (a_i, N_i) for $i < \delta$ and $f_{j,i} : N_j \to N_i$ for $j < i < \delta$ such that:

(1) $f_{k,i} = f_{j,i} \circ f_{k,j}$ for all $k < j < i$.
(2) $\mathbf{gtp}(a_i/M_i, N_i) = p_i$.
(3) $f_{j,i} \restriction_{M_j} = id_{M_j}$.
(4) $f_{j,i}(a_j) = a_i$.

The notion of coherent sequence of types first appeared in [GV06, 2.12], Here we use the version in [MA20, 3.14] that avoids the use of a monster model.

Fact 2.9. [Bal09, 12.3] Let δ be a limit ordinal and $\langle M_i \in K \mid i \leq \delta \rangle$ increasing continuous, and $\langle p_i \in gS(M_i) \mid i < \delta \rangle$ a coherent sequence of types. Then there is $p \in gS(M_\delta)$ an upper bound of $\langle p_i \in gS(M_i) \mid i < \delta \rangle$, where the order is the one from Definition 2.5(5).

Fact 2.10. [Bal09, 11.3(2)] Let δ be a limit ordinal, $\langle M_i \in K \mid i \leq \delta \rangle$ increasing continuous, and $\langle p_i \in gS(M_i) \mid i < \delta \rangle$ a sequence of types with upper bound $p \in gS(M_\delta)$. Then there are $\langle N_i \mid i \leq \delta \rangle$ and $\langle f_{j,i} \mid j < i \rangle$ that witness $\langle p_i \in gS(M_i) \mid i \leq \delta \rangle$ being a coherent sequence.

Definition 2.11. [She01, 0.22(2)] Let $\mu > \lambda$. $N \in K_\mu$ is *saturated in* μ *above* λ if for all $M \leq_K N$, $\lambda \leq \|M\| < \mu$, N realizes $gS(M)$.

Definition 2.12. [She01, 0.26(1)] Let $\mu > \lambda$. $N \in K_\mu$ is *homogeneous in* μ *for* λ if for all $M_1 \leq_K N$, $M_1 \leq_K M_2 \in K_\lambda$, $\lambda \leq \|M_1\| \leq \|M_2\| < \mu$, there is K-embedding $f : M_2 \to N$ above M_1.

Fact 2.13. [She01, 0.26(1)] Let $\mu > \lambda$. If K_λ has AP then $M \in K_\mu$ is saturated over μ for λ if and only if M is homogeneous over μ for λ.

Definition 2.14. [She71a] For a cardinal λ,

$$\text{ded } \lambda := \sup\{\kappa \mid \exists \text{ a regular } \mu \text{ and a tree } T \text{ with } \leq \lambda \text{ nodes and } \kappa \text{ branches of length } \mu, |T| = \kappa\}.$$

Fact 2.15. [She78, II.4.11] Let T be a complete first order theory and ϕ a formula in its language. λ is an infinite cardinal such that $2^\lambda > \text{ded } \lambda$. The following are equivalent:

(1) ϕ has the independence property.
(2) $|S_\phi(A)| > \text{ded } |A|$ for some infinite set A, $|A| = \lambda$.
(3) $|S_\phi(A)| = 2^{|A|}$ for some infinite set A, $|A| = \lambda$.

Fact 2.16. [She78, II.4.12] Let T be a complete theory in countable language, and $f_T(\lambda) := \sup\{|S(M)| \mid M \models T, \|M\| = \lambda\}$. Then $f_T(\lambda)$ is exactly one of: λ, $\lambda + 2^{\aleph_0}$, $\lambda^{\aleph_0}$, $\text{ded } \lambda$, $(\text{ded } \lambda)^{\aleph_0}$ or 2^λ. See also [Kei76].

It is reasonable to propose the following definition:

Definition 2.17. Let K be an AEC, $\lambda \geq LS(K)$. K_λ has NIP if for all $M \in K_\lambda$, $|gS(M)| \leq \text{ded } \lambda$.

At present it is unclear that we have discovered the "correct" notion. In fact, it is plausible that there are several different notions that are equivalent when K is an elementary class, but distinct for some non-elementary K. One weakness of our definition is that unlike the corresponding first order notion, it is probably not absolute.

Grossberg raised the following question:

Question 2.18. Is there an equivalent notion which does not rely on extra set theoretic assumptions. (at least for AECs K with $LS(K) = \aleph_0$ which are also $PC_{\aleph_0}$)?

Fact 2.19. [JS13, 2.5.8] Assume K has JEP, AP and NMM. Suppose there is $S^{bs} \subseteq gS$ family of types on K satisfying only (Density), (Invariance), and for all $M \in K_\lambda$, $|S^{bs}(M)| \leq \lambda^+$. See Definitions 3.1 and 3.3.

(1) If $\langle M_\alpha \in K_\lambda \mid \alpha < \lambda^+ \rangle$ is increasing and continuous, and there is a stationary set $S \subseteq \lambda^+$ such that for every $\alpha \in S$ and every model N, with $M_\alpha \leq_K N$, there is a type $p \in S^{bs}(M_\alpha)$ which is realized in M_{λ^+} and in N, then M_{λ^+} is saturated in λ^+ above λ.
(2) For all $M \in K_\lambda$, $|gS(M)| \leq \lambda^+$.

The following is an example of an AEC satisfying NIP that is not elementary or stable.

Example 2.20. [JS13, 2.2.4] Let λ be a cardinal. Let P be a family of λ^+ subsets of λ. Let $\tau := \{R_\alpha : \alpha < \lambda\}$ where each R_α is an unary predicate. Let K be the class of models M for τ such that for each $a \in |M|$, $\{\alpha \in \lambda \mid M \models R_\alpha(a)\} \in P$. Note that K is not elementary. Let $\leq_K$ be the substructure relation on K. The trivial λ-frame on K_λ satisfies the axioms of a semi-good λ-frame [JS13, 2.1.3], so in particular by Fact 2.19 K_λ satisfies NIP. On the other hand, it is unstable.

The next is an algebraic example of an AEC that satisfies NIP and is not elementary or stable.

Example 2.21. (ded λ = (ded $\lambda)^{\aleph_0}$) Let K be the class of real closed fields, and $F \leq_K L$ if and only if $F \preceq L$ and L/F is a normal extension, so $(K, \leq_K)$ is not elementary. Since $(K, \preceq)$ is NIP but unstable, the number of $L_{\omega,\omega}$ syntactic types over $M \in K_\lambda$, which are orbits of $\mathrm{Aut}_M(\mathfrak{C})$, coincide with Galois types $gS(M)$. The number of types is bounded by ded λ = (ded $\lambda)^{\aleph_0}$ but strictly more than λ.

Definition 2.22. [She09d, VI.1.12(1)] We say S_* is a $\leq_{K_\lambda}$-*type-kind* when:

(1) S_* is a function with domain K_λ.
(2) $S_*(M) \subseteq gS(M)$ for all $M \in K_\lambda$.
(3) $S_*(M)$ commutes with isomorphisms.

Definition 2.23. [She09d, VI.2.9]

(1) For $M \in K$ and $\Gamma \subseteq gS(M)$, Γ is *inevitable* if for all $N >_K M$ there is $a \in |N| - |M|$ with $\mathbf{gtp}(a/M, N) \in \Gamma$.
(2) For $M \in K$ and $\Gamma \subseteq gS(M)$, Γ is S_*-*inevitable* if for all $N >_K M$, if there is $p \in S_*(M)$ realized in N then there is $q \in \Gamma$ realized in N.

Definition 2.24. [She09d, VI.1.12(2)] For $\leq_{K_\lambda}$-type-kinds S_1 and S_2, say S_1 is *hereditarily in* S_2 when: for $M \leq_K N$ and $p \in S_2(N)$ we have $p \restriction_M \in S_1(M) \implies p \in S_1(N)$.

Definition 2.25. Let $M \in K_\lambda$. $p \in gS(M)$ is $<\mu$*-minimal* if for all $M \leq N \in K_\lambda$, $|\{q \in gS(N) : q \restriction_M = p\}| < \mu$.

$$S^{<\mu-minimal}(M) := \{p \in gS(M) \mid p \text{ is } <\mu\text{-minimal}\}.$$

Remark 2.26. $S^{<\mu-minimal}$ and $S^{\lambda-al}$ (defined in Lemma 3.13) are hereditarily in gS.

The following principle known as the weak diamond was introduced by Devlin and Shelah [DS78].

Definition 2.27. Let $S \subseteq \lambda^+$ be a stationary set. $\Phi^2_{\lambda^+}(S)$ holds if and only if for all $F : (2^\lambda)^{<\lambda^+} \to 2$ there exists $g : \lambda^+ \to 2$ such that for all $f : \lambda^+ \to 2^\lambda$ the set $\{\alpha \in S : F(f \restriction_\alpha) = g(\alpha)\}$ is stationary.

Fact 2.28. [DS78]

(1) $2^\lambda < 2^{\lambda^+}$ if and only if $\Phi^2_{\lambda^+}(\lambda^+)$ holds.
(2) $\Phi^2_{\lambda^+}(S)$ holds for a stationary set $S \subseteq \lambda^+$ if and only if $\forall F : (2 \times 2 \times \lambda^+)^{<\lambda^+} \to 2\ \exists g : \lambda^+ \to 2$ such that $\forall \eta \in 2^{\lambda^+} \forall \nu \in 2^{\lambda^+} \forall h : \lambda^+ \to \lambda^+$ the set $\{\alpha \in S : F(\eta \restriction_\alpha, \nu \restriction_\alpha, h \restriction_\alpha) = g(\alpha)\}$ is stationary.
(3) If $\Phi^2_{\lambda^+}(\lambda^+)$ holds then there exists $\{S_i \subseteq \lambda^+ : i < \lambda^+\}$ pairwise disjoint stationary sets such that $\Phi^2_{\lambda^+}(S_i)$ for each $i < \lambda^+$.

Fact 2.29. [She09d, VI.2.18] ($2^\lambda < 2^{\lambda^+}$) Assume K has amalgamation and no maximal model in λ. If

(1) S_* is $\leq_{K_\lambda}$-type-kind and hereditary,
(2) $S_* \subseteq gS^{<\lambda^+-min}$, and
(3) There is $M \in K_\lambda$ such that:
 (a) $|gS_*(M)| \geq \lambda^+$, and
 (b) if $M \leq_K N \in K_\lambda$, no subset of $S_*(N)$ of size $\leq \lambda$ is S_*-inevitable,

then $I(\lambda^+, K) = 2^{\lambda^+}$.

Fact 2.30. [She09d, VI.2.11(2)][2] For every $M \in K_\lambda$ we have $|S_*(M)| \leq \lambda$ when:

(1) K has AP in λ.
(2) S_* is a hereditary $\leq_{K_\lambda}$-type-kind in gS.
(3) For every $M \in K_\lambda$ there is an S_*-inevitable $\Gamma_M \subseteq gS(M)$ of cardinality $\leq \lambda$.

3. The w*-good frame

In this section we define w*-good frames, and show that K_λ has NIP if and only if K has a w*-good λ-grame under additional assumptions.

[2] A complete argument of this result does not appear in [She09d].

Definition 3.1. [She09c, III.0] Let $\lambda < \mu$, where λ is a cardinal, and μ is a cardinal or ∞. A *pre-$[\lambda,\mu)$-frame* is a triple $\mathfrak{s} = (K, \mathrel{\unicode{x2AEB}}, S^{bs})$ such that:

(1) K is an AEC with $\lambda \geq LS(K)$ and $K_\lambda \neq \emptyset$.
(2) $S^{bs} \subseteq \bigcup_{M \in K_{[\lambda,\mu)}} gS(M)$. Let $S^{bs}(M) := gS(M) \cap S^{bs}$. Types in this family are called *basic types*.
(3) $\mathrel{\unicode{x2AEB}}$ is a relation on quadruples (M_0, M_1, a, N), where $M_0 \leq_K M_1 \leq N$, $a \in |N|$ and $M_0, M_1, N \in K_{[\lambda,\mu)}$. We write $a \underset{M_0}{\overset{N}{\downarrow}} M_1$, or we say $\mathbf{gtp}(a/M_1, N)$ does not fork over M_0 when the relation $\mathrel{\unicode{x2AEB}}$ holds for (M_0, M_1, a, N).
(4) (Invariance) If $f : N \cong N'$ and $a \underset{M_0}{\overset{N}{\downarrow}} M_1$, then $f(a) \underset{f[M_0]}{\overset{N'}{\downarrow}} f[M_1]$. If $\mathbf{gtp}(a/M_1, N) \in S^{bs}(M_1)$, then $\mathbf{gtp}(f(a)/f[M_1], N') \in S^{bs}(f[M_1])$.
(5) (Monotonicity) If $a \underset{M_0}{\overset{N}{\downarrow}} M_1$ and $M_0 \leq_K M_0' \leq_K M_1' \leq_K M_1 \leq_K N' \leq_K N \leq_K N''$ with $N'' \in K_{[\lambda,\mu)}$ and $a \in |N'|$, then $a \underset{M_0'}{\overset{N'}{\downarrow}} M_1'$ and $a \underset{M_0'}{\overset{N''}{\downarrow}} M_1'$.
(6) (Non-forking Types are Basic) If $a \underset{M}{\overset{N}{\downarrow}} M$ then $\mathbf{gtp}(a/M, N) \in S^{bs}(M)$.

Definition 3.2. [MA20, 3.6] A pre-$[\lambda,\mu)$-frame $\mathfrak{s} = (K, \mathrel{\unicode{x2AEB}}, S^{bs})$ is a *w-good frame* if:

(1) $K_{[\lambda,\mu)}$ has AP, JEP and NMM.
(2) (Weak Density) For all $M <_K N \in K_\lambda$, there is $a \in |N| - |M|$ and $M' \leq N' \in K_{[lambda,\mu)}$ such that $(a, M, N) \leq (a, M', N')$ and $\mathbf{gtp}(a/M', N') \in S^{bs}(M')$.
(3) (Existence of Non-Forking Extension) If $p \in S^{bs}(M)$ and $M \leq_K N$, then there is $q \in S^{bs}(N)$ extending p which does not fork over M.
(4) (Uniqueness) If $M \leq_K N$ both in $K_{[\lambda,\mu)}$, $p, q \in S^{bs}(N)$ both do not fork over M, and $p\restriction_M = q\restriction_M$, then $p = q$.
(5) (Strong Continuity[3]) If $\delta < \mu$ a limit ordinal, $\langle M_i \mid i \leq \delta \rangle$ increasing and continuous, $\langle p_i \in S^{bs}(M_i) \mid i < \delta \rangle$, and $i < j < \delta$ implies $p_j \restriction M_i = p_i$, and $p_\delta \in S(M_\delta)$ is an upper bound for $\langle p_i \mid i < \delta \rangle$, then $p \in S^{bs}(M_\delta)$. Moreover, if each p_i does not fork over M_0 then neither does p_δ.

Definition 3.3. A pre-$[\lambda,\mu)$-frame $\mathfrak{s} = (K, \mathrel{\unicode{x2AEB}}, S^{bs})$ is a *w^*-good* frame if $\mathfrak{s}$ satisfies:

(1) $K_{[\lambda,\mu)}$ has AP, JEP and NMM.

[3]This was called just continuity in [MA20]. The author would like to thank Marcos Mazari-Armida for pointing out that the continuity axiom of a good frame requires only the moreover part.

(2) (Uniqueness). See Definition 3.2.
(3) (Basic NIP) For all $M \in K_{[\lambda,\mu)}$ $|S^{bs}(M)| \leq \text{ded } \|M\|$.
(4) (Few Non-Basic Types) For all $M \in K_{[\lambda,\mu)}$, $|gS(M) - S^{bs}(M)| \leq \lambda$.
(5) (Continuity[4]) If $\delta < \mu$ a limit ordinal, $\langle M_i \mid i \leq \delta \rangle$ increasing and continuous, $\langle p_i \in S^{bs}(M_i) \mid i < \delta \rangle$, and $i < j < \delta$ implies $p_j \restriction_{M_i} = p_i$, and $p_\delta \in gS(M_\delta)$ is an upper bound for $\langle p_i \mid i < \delta \rangle$. If each p_i does not fork over M_0 then $p_\delta \in S^{bs}(M_\delta)$ and p_δ also does not fork over M_0.
(6) (Transitivity) if $p \in S^{bs}(M_2)$ does not fork over $M_1 \leq_K M_2$, and $p \restriction_{M_1}$ does not fork over $M_0 \leq_K M_1$, then p does not fork over M_0.

Although the author cannot find a proof or counterexample, w-good and w*-good frames are likely to be incomparable.

Remark 3.4. (Continuity) is weaker than (Strong Continuity). Without not forking over M_0 one cannot deduce that $p_\delta \in S^{bs}(M_\delta)$.

Remark 3.5. In a w-good frame (Transitivity) is implied by several other properties including (Existence of Non-Forking Extension). For a w*-good frame, where (Existence of Non-Forking Extension) does not hold in general, we need to explicitly include (Transitivity) as an axiom.

Definition 3.6. When $\mu = \lambda^+$ in the previous definitions, we say $\mathfrak{s}$ is a pre-/w-good/w*-good λ-frame.

From now on we build a w*-good λ-frame on K assuming the following:

Hypothesis 3.7 ($2^{\lambda^+} > 2^\lambda$)**.** We fix K an AEC and a cardinal $\lambda \geq LS(K)$ such that K is categorical in λ. Further more $1 \leq I(\lambda^+, K) < 2^{\lambda^+}$, and K_λ has NIP.

As K is categorical in λ, then K has λ-AP by the following fact, which appeared in [She87, 3.5] first, and a clearer proof can be found in [Gro02, 4.3]. λ-JEP follows from categoricity, and λ-NMM follows from categoricity and $K_{\lambda^+} \neq \emptyset$.

Fact 3.8. [She87, 3.5] ($2^\lambda < 2^{\lambda^+}$) If $I(\lambda, K) = 1 \leq I(\lambda^+, K) < 2^{\lambda^+}$, then K has the λ-AP.

Definition 3.9. $p = \mathbf{gtp}(a/M, N)$ has the extension property if for every K-embedding $f : M \to M_1 \in K_\lambda$ there is $q \in gS(M_1)$ extending $f(p)$.

Definition 3.10. $p = \mathbf{gtp}(a/M, N)$ is λ-*unique*[5]. if

(1) $p = \mathbf{gtp}(a/M, N)$ has the extension property, and

[4] This is the continuity axiom for good frames.

[5] This notion was first introduced by Shelah in [She75, 6.1], called minimal types there. Note that this is a different notion from the minimal types of [She01]. These types are also called *quasiminimal types* in the literature, see for example [Zil05] and [Les05]

(2) for every $M \leq_K M' \in K_\lambda$, p has at most one extension $q \in gS(M')$ with the extension property.

Fact 3.11. [She09d, VI.2.5(2B)] If K_λ has AP and $\lambda \geq LS(K)$, $\mathbf{gtp}(a, M, N)$ has $\geq \lambda^+$ realizations in some extension of M (necessarily in $K_{\geq \lambda^+}$) if and only if $\mathbf{gtp}(a/M, N)$ has the extension property.

Now we define the w*-good λ-frame.

Definition 3.12. The preframe $\mathfrak{s}_{\lambda-unq}$ is defined such that:

(1) $S^{bs}(M) := \{p = \mathbf{gtp}(a/M, N) \mid p$ has the extension property$\}$.
(2) $p = \mathbf{gtp}(a/M, N) \in S^{bs}(M)$ does not fork over $M_0 \leq_K M$ if $p \restriction_{M_0}$ is λ-unique.

Lemma 3.13. $S^{\lambda-al}(M) := \{p \in gS(M) \mid p$ has $\leq \lambda$-many realizations$\}$ satisfies $|S^{\lambda-al}(M)| \leq \lambda$. By realizations we mean realizations in any $\leq_K$-extension of M in K_{λ^+}. So $\mathfrak{s}_{\lambda-unq}$ satisfies (Few Non-Basic Types).

Proof. Suppose not, i.e. $|S^{\lambda-al}(M)| \geq \lambda^+$.
Claim: There is no $\Gamma \subseteq S^{\lambda-al}(M)$, $|\Gamma| \leq \lambda$ that is inevitable.

Otherwise, suppose there exists such Γ. By Fact 2.30, taking S_* to be gS, and Γ_M to be Γ, we have $|gS(M)| \leq \lambda$, so in particular $|S^{\lambda-al}(M)| \leq \lambda$, contradiction.

Now by the claim and Fact 2.29, taking S_* there to be $S^{\lambda-al}$ and μ there to be λ^+, we have $I(\lambda^+, K) = 2^{\lambda^+}$, contradiction. $\square$

Thus from now on in this section we also assume $|S^{\lambda-al}(M)| \leq \lambda$.

Lemma 3.14. $\mathfrak{s}_{\lambda-unq}$ satisfies the following properties in Definitions 3.1, 3.2 and 3.3:

(1) (Invariance).
(2) (Monotonicity).
(3) (Non-Forking Types are Basic).
(4) AP, JEP and NMM.
(5) (Basic NIP).
(6) (Uniqueness).
(7) (Transitivity).

Proof. Easy. We prove (Transitivity) as an example. Suppose $p \in S^{bs}(N)$ does not fork over $M_1 \leq_K N$, and $p \restriction_{M_1}$ does not fork over $M_0 \leq_K M_1$. Then $(p \restriction_{M_1}) \restriction_{M_0}$ is λ-unique, i.e. $p \restriction_{M_0}$ is. Thus p does not fork over M_0. $\square$

The following property is essential for the next lemma.

Definition 3.15. A type family S_* is λ-compact if for every limit ordinal $\delta < \lambda^+$, for every $\langle M_i \in K_\lambda : i < \delta\rangle$ an increasing continuous chain and for every coherent sequence of types $\langle p_i \in S_*(M_i) : i < \delta\rangle$, there is an upper bound $p \in S_*(\bigcup_{i<\delta} M_i)$ to the sequence such that $\langle p_i \in S_*(M_i) : i < \delta + 1\rangle$ is a coherent sequence.

For certain results in this paper we need to assume that the basic types (i.e. those with the extension property) is λ-compact, which, for example, holds for AECs with the disjoint amalgamation property, where every type has the extension property. Many classes of modules have the disjoint amalgamation property. See [MAR23, 2.10] and [BET07, 2.2].

Lemma 3.16 (ded $\lambda = \lambda^+ < 2^\lambda$)**.** Suppose that S^{bs} is λ-compact. If $p \in S^{bs}(M)$, then there is $N \geq_K M$ and $q \in S^{bs}(N)$ extending p that does not fork over N. In particular, for any $N' \geq_K N$ there is unique $q' \in gS(N')$ extending q that does not fork over N.

Proof. It suffices to show that there is a λ-unique type above any basic type. By Fact 2.19 let $\mathfrak{C} \in K_{\lambda^+}$ be saturated in λ^+ over λ. It is also homogeneous in λ^+ over λ by Fact 2.13. Let $(a, M, N) \in K^3_\lambda$ such that $\mathbf{gtp}(a/M, N)$ has the extension property and there is no λ-unique type above $\mathbf{gtp}(a/M, N)$. Build $(a_\eta, M_\eta, N_\eta) \in K^3_\lambda$ for $\eta \in^{<\lambda} 2$ and $h_{\eta,\nu}$ for $\eta < \nu \in^{<\lambda} 2$ such that:

(1) $(a_{\langle\rangle}, M_{\langle\rangle}, N_{\langle\rangle}) = (a, M, N)$.
(2) $(a_\eta, M_\eta, N_\eta) \leq_{h_{\eta,\nu}} (a_\nu, M_\nu, N_\nu)$ for $\eta < \nu$.
(3) $h_{\eta,\rho} = h_{\nu,\rho} \circ h_{\eta,\nu}$ for $\eta < \nu < \rho$.
(4) $M_{\eta^\frown 0} = M_{\eta^\frown 1}$, $N_{\eta^\frown 0} = N_{\eta^\frown 1}$, and $h_{\eta,\eta^\frown 0} \restriction M_\eta = h_{\eta,\eta^\frown 1} \restriction M_\eta$.
(5) $\mathbf{gtp}(a_{\eta^\frown 0}, M_{\eta^\frown 0}, N_{\eta^\frown 0}) \neq \mathbf{gtp}(a_{\eta^\frown 1}, M_{\eta^\frown 1}, N_{\eta^\frown 1})$, both having λ^+-many realizations.
(6) If $\eta \in^\delta 2$ for δ a limit ordinal, take M_η and N_η to be directed colimits.

Construction: Base case and limit case are clear. At successor stage use non-λ-uniqueness to get two distinct extensions, each having λ^+-many realizations.
Enough: Let $M \leq_K \mathfrak{C} \in K_{\lambda^+}$ be saturated over λ. Build $g_\eta : M_\eta \to \mathfrak{C}$ for $\eta \in {}^\lambda 2$ such that:

(1) $g_\nu \circ h_{\eta,\nu} = g_\eta$ for $\nu < \eta$.
(2) $g_{\eta^\frown 0} = g_{\eta^\frown 1}$

This is possible: We carry out the construction by induction on the $\ell(\eta)$, the length of η. For the base case take $g_{\langle\rangle}$ to be inclusion $M \leq_K \mathfrak{C}$. At limit use the universal property of M_η as a directed colimit. For the successor case, for η of length $\alpha = \beta + 1$, suppose we have g_η.

(1)
$$\begin{tikzcd}
\mathfrak{C} & M''_{\eta^\frown 0} \arrow[l, "j"'] & M'_{\eta^\frown 0} \arrow[l, "\cong_h"'] \arrow[r, "\cong_g"] & M_{\eta^\frown 0} \\
& g_\eta[M_\eta] \arrow[ul, "id"] \arrow[u, "id"] & M_\eta \arrow[l, "\cong_{g_\eta}"'] \arrow[u, "id"] \arrow[r, "\cong_{h_{\eta,\eta^\frown 0}}"] & h_{\eta,\eta^\frown 0}[M_\eta] \arrow[u, "id"']
\end{tikzcd}$$

Use basic extension to obtain the right square and g, and then obtain the middle square and h. Finally the left triangle is by saturation of $\mathfrak{C}$. Now define $g_{\eta^\frown 0} = g_{\eta^\frown 1}$ to be the composition of the top row from right to left.

This is enough: For each branch $\eta \in {}^{\lambda}2$, take directed colimit to obtain (a_η, M_η, N_η). Obtain $f_\eta : M_\eta \to \mathfrak{C}$ by the universal property of colimits such that $f_\eta \circ h_{\nu,\eta} = g_\nu$ for all $\nu < \eta$, and obtain $f'_\eta : N_\eta \to \mathfrak{C}$ extending f_η by saturation over λ. Since each $f'_\eta(a_\eta) \in |\mathfrak{C}|$, but $\|\mathfrak{C}\| = \text{ded } \lambda < 2^\lambda$, there must be $\eta, \nu \in^\lambda 2$ such that $f'_\eta(a_\eta) = f'_\nu(a_\nu)$. Let $\alpha < \lambda$ be the least such that $\eta(\alpha) \neq \nu(\alpha)$. Without loss of generality say $\eta(\alpha) = 0$ and $\nu(\alpha) = 1$. Then the following diagram commutes:

(2)
$$\begin{tikzcd}
N_{\eta\restriction_\alpha{}^\frown 0} \arrow[r, "f'_\eta \circ h_{\eta\restriction_\alpha{}^\frown 0,\eta}"] & \mathfrak{C} \\
M_{\eta\restriction_\alpha{}^\frown 0} \arrow[u, "id"] \arrow[r, "id"] & N_{\eta\restriction_\alpha{}^\frown 1} \arrow[u, "f'_\nu \circ h_{\eta\restriction_\alpha{}^\frown 1,\nu}"]
\end{tikzcd}$$

with $f'_\eta \circ h_{\eta\restriction_\alpha{}^\frown 0,\eta}(a_{\eta\restriction_\alpha{}^\frown 0}) = f'_\nu \circ h_{\eta\restriction_\alpha{}^\frown 1,\nu}(a_{\eta\restriction_\alpha{}^\frown 1})$ since $f'_\eta(a_\eta) = f'_\nu(a_\nu)$, contradicting requirement (5) of the construction. $\square$

Remark 3.17. The proof of Lemma 3.16 is along the argument of Mazari-Armida in [MA20, 4.13] and [She09d, VI.2.25], and the difference is that there the saturated model over λ lies in $K_{\lambda^{++}}$. For completeness we included all the details.

Question 3.18. Lemma 3.16 is a weaker form of (Existence of Non-Forking Extension). Is it possible to obtain (Existence of Non-Forking Extension) in its full strength, by perhaps considering another family of basic types and non-forking relation? One could imitate the w-good λ-frame in [MA20] and use λ-unique types as basic ones, and then Lemma 3.16 gives a proof of (Weak Density). However, then it is hard to show that having such a frame implies NIP.

The following definition is [She99, 1.8], which is defined for types of any finite length. Here we only need it for length 1. Thus we use the version from [Bal09, 11.4(1)].

Definition 3.19. (1) K is *(κ, λ)-local* if for every increasing continuous chain $M = \bigcup_{i<\kappa} M_i$ with $\|M\| = \lambda$ and for any $p, q \in gS(M)$: if $p \restriction_{M_i} = q \restriction_{M_i}$ for all i then $p = q$.

(2) K is $(<\kappa,\lambda)$-*local* if K is (μ,λ)-local for all $\mu<\kappa$.

Lemma 3.20. If K is $(<\lambda^+,\lambda)$-local, then $\mathfrak{s}_{\lambda-unq}$ has (Continuity).

Proof. Let M_i, $i<\delta$ be increasing continuous. $p_i \in S^{bs}(M_i)$ increasing and for $i<j<\delta$ we have $p_j \restriction_{M_i} = p_i$, and p_δ upper bound. Suppose p_δ has $\leq \lambda$-many realizations. Then there is a set S of cardinality λ^+ of realizations of p_0, such that for each $a\in S$, by locality there is $i<\delta$ such that a realizes p_i but not p_{i+1}. By pigeonhole principle for some $i<\delta$ there are λ^+-many realizations of p_i that are not realizations of p_{i+1}. Since there are $\leq\lambda$-many types in $S(M_{i+1})$ that have $\leq\lambda$-many realizations, there must be another type in $S(M_{i+1})$ with λ^+ realizations distinct from p_{i+1}, which contradicts λ-uniqueness of p_{i+1}.

For the moreover part, if p_0 does not fork over M_0, so $p_0 = p_\delta\restriction_{M_0}$ is λ-unique, i.e. p_δ does not fork over M_0. □

Theorem 3.21 ($2^{\lambda^+} > 2^\lambda$)**.** Let K be an AEC categorical in $\lambda \geq LS(K)$, and $1\leq I(\lambda^+,K) < 2^{\lambda^+}$. K_λ has NIP if and only if there is a w*-good λ-frame on K except possibly without (Continuity). Moreover,

1. (ded $\lambda=\lambda^+<2^\lambda$) If $\mathfrak{s}_{\lambda-unq}$ is λ-compact, then the w*-good frame satisfies in addition that if $p\in S^{bs}(M)$, then there is $N\geq_K M$ and $q\in S^{bs}(N)$ extending p that does not fork over N. In particular, for any $N'\geq_K N$ there is $q'\in gS(N')$ extending q that does not fork over N.
2. if K is $(<\lambda^+,\lambda)$-local, then $\mathfrak{s}_{\lambda-unq}$ has (Continuity).

Proof. The moreover part follows from Lemma 3.16. □

4. Syntactic independence property

In this section we assume tameness, and use Galois Morleyization to show that the negation of NIP leads to being able to encode subsets, as a parallel of first order independence property.

Hypothesis 4.1. Let κ be an infinite cardinal and K an AEC. Let $\tau = L(K)$ be its underlying language.

We first extend the definition of Galois types to longer lengths and set-valued domains.

Definition 4.2. (1) $K^3 := \{(\bar a, A, N) \mid N\in K, A\subseteq |N|, \bar a \text{ is a sequence from } |N|\}$.

(2) For $(\bar a_0, A, N_0), (\bar a_1, A, N_1)\in K^3$, $(\bar a_0, A, N_0)E_{at}(\bar a_1, A, N_1)$ if there are $N\in K$, $f_0 : N_0 \to_A N$, and $f_1 : N_1\to_A N$ K-embeddings such that $f_0(\bar a_0) = f_1(\bar a_1)$, $f_0\restriction_A = f_1\restriction_A$.

(3) E is the transitive closure of E_{at}.

(4) For $(\bar{a}, A, N) \in K^3$, *the Galois type of* $\bar{a}$ *over* A *in* N is $\mathbf{gtp}(a/A, N) := [(a, A, N)]_E$.
(5) For $N \in K$ and $A \subseteq |N|$, α an ordinal or ∞, $gS^{<\alpha}(A; N) := \{\mathbf{gtp}(\bar{a}/A, N) \mid (\bar{a}, A, N) \in K^3 \text{ and } \bar{a} \in^{<\alpha} |N|\}$. $gS^{\alpha}(A; N)$ is defined similarly.

Remark 4.3. In the case where $A = |M|$ for $M \in K$, $\bigcup_{N \geq_K M} gS^1(|M|, N)$ is what we defined as $gS(M)$ in Definition 2.5.

The following technique first appeared in [Vas16c], which allows one to work with Galois types in a syntactic way.

Definition 4.4. Let κ be an infinite cardinal and K an AEC. The $(< \kappa)$-*Galois Morleyization* of K is $\hat{K}$, an AEC (except that the language might not be finitary) in a $(< \kappa)$-ary language $\hat{\tau}$ extending τ such that:

(1) The structures and the substructure relation $\leq_{\hat{K}}$ in $\hat{K}$ are the same as K.
(2) For each $p \in gS^{<\kappa}(\emptyset)$, there is a predicate of the same length $R_p \in \hat{\tau}$. For each $M \in K$ and $\bar{a} \in |M|$, define $M \models R_p[\bar{a}]$ if and only if $\mathbf{gtp}(\bar{a}/\emptyset, M) = p$. By extension, one can interpret quantifier-free $L_{\kappa,\kappa}(\hat{\tau})$ formulas.
(3) The $(< \kappa)$-syntactic type of $\bar{a} \in^{<\kappa} |M|$ over $A \subseteq |M|$ is $\mathbf{tp}_{\text{qf-}L_{\kappa,\kappa}(\hat{\tau})}(\bar{a}/A, M)$, the set of all quantifier-free $L_{\kappa,\kappa}(\hat{\tau})$ formulas with parameters from A that $\bar{a}$ satisfies. For a particular quantifier-free $L_{\kappa,\kappa}(\hat{\tau})$-formula $\phi(\bar{x}, \bar{y})$, $\mathbf{tp}_{\phi}(\bar{b}/A, M) := \{\phi(\bar{x}, \bar{a}) \mid \bar{a} \in A, M \models \phi(\bar{b}, \bar{a})\}$.
(4) For $M \in K$ and $A \subseteq |M|$, $S^{<\alpha}_{\text{qf}-L_{\kappa,\kappa}(\hat{\tau})}(A; M) := \{\mathbf{tp}_{\text{qf-}L_{\kappa,\kappa}(\hat{\tau})}(\bar{b}/A, M) \mid \bar{b} \in^{<\alpha} |M|\}$

Remark 4.5. There are $\leq 2^{<(LS(K)^+ + \kappa)}$ formulas in $\hat{\tau}$.

Definition 4.6. For a theory T in first order logic, and Γ a set of T-types, τ a language contained in the language of T, let $EC(T, \Gamma)$ denote the class of models of T omitting all types in Γ. Let $PC(T, \Gamma, \tau)$ denote the class of models of T omitting all types in Γ as τ-structures.

Fact 4.7. [Vas16c, 3.18(2)] Under the notation of the previous definition, K is $(< \kappa)$-tame if and only if for each ordinal α, $M \in K$, $A \subseteq M$, $\mathbf{gtp}(\bar{b}/A, M) \mapsto \mathbf{tp}_{\text{qf-}L_{\kappa,\kappa}(\hat{\tau})}(\bar{b}/A, M)$ from $gS^{\alpha}(A; M)$ to $S^{\alpha}_{\text{qf}-L_{\kappa,\kappa}(\hat{\tau})}(A; M)$ is bijective.

Notation 4.8. For any formula φ and a condition i, φ^i means φ itself when i holds, and $\neg\varphi$ otherwise. For example, at the end of the proof of the next theorem, the formula is $\phi(c_i, x)$ and the condition is $i \in w$. When $i \in w$ holds, $\phi(c_i, x)^{i \in w}$ is $\phi(c_i, x)$. When $i \notin w$, $\phi(c_i, x)^{i \in w}$ is $\neg\phi(c_i, x)$.

Definition 4.9. For T a first order theory, Γ a set of T-types, let $EC(T, \Gamma)$ denote the class of T-models that omit all types in Γ. If moreover τ is a language such that all of its symbols appear in the language of T, let $PC(T, \Gamma, \tau)$ denote the class of T models omitting each type in Γ interpreted as τ-structures.

Theorem 4.10. Suppose K is $(<\aleph_0)$-tame, $M \in K$, $C \subseteq |M|$, $\lambda := |C| \geq \beth_3(LS(K))$ and $(\text{ded }\lambda)^{2^{LS(K)}} = \text{ded }\lambda$. Suppose $|gS^1(C;M)| > \text{ded }\lambda$. Then there is $N \in K$, $\langle \bar{a}_n \in^m |N| \mid n < \omega\rangle$ and ϕ in the language of Galois Morleyization such that for every $w \subseteq \omega$ there is $b_w \in |N|$ such that for all $i < \omega$,

$$N \models \phi(\bar{a}_i, b_w) \iff i \in w$$

Proof. Let $\hat{K}$ be the $(<\aleph_0)$ Galois Morleyization of K. Note that both classes have the same Galois types. By Shelah's Presentation Theorem $\hat{K} = PC(T, \Gamma, \hat{\tau})$ with $|T| \leq 2^{LS(K)}$, with the language of T containing $\hat{\tau}$. Then by tameness and the previous fact $|S^1_{\text{qf}-L_{\omega,\omega}(\hat{\tau})}(C;M)| > \text{ded }\lambda$, so for some quantifier-free formula $\phi(\bar{y}, x)$ in $L_{\omega,\omega}(\hat{\tau})$ with $|S_\phi(C;M)| > \text{ded }\lambda$, since there are $\leq 2^{LS(K)}$-many quantifier-free $L_{\omega,\omega}(\hat{\tau})$-formulas.

Without loss of generality $C = \lambda = |C|$. Let $\mu := (\text{ded }\lambda)^+$. For notational simplicity we view $S_\phi(C;M)$ as S, a family of subsets of ${}^{\ell(\bar{y})}C$, where

$$A \in S \iff \{\phi(\bar{a}, x) \mid \bar{a} \in A\} \in S_\phi(C).$$

We also assume $\bar{y}$ has length 1. The proof for other cases is similar.

Claim: For all $\alpha < \lambda$, if $|\{A \cap \alpha \mid A \in S\}| \geq \mu$, then $\alpha \geq (\beth_2(LS(K)))^+$.
Proof of Claim: Suppose there is $\alpha < \lambda$, $|\{A \cap \alpha \mid A \in S\}| \geq \mu$. Since $\{A \cap \alpha \mid A \in S\}$ is the set of branches of the a subtree of ${}^{<\alpha}2$, $\text{ded }\lambda < \mu \leq \text{ded }|{}^{<\alpha}2| \leq \text{ded }2^{|\alpha|}$, so $2^{|\alpha|} > \lambda \geq \beth_3(LS(K))$, so $|\alpha| > \beth_2(LS(K))$. Thus the claim holds.

We may assume $\lambda > \beth_2(LS(K))$ and for all $\alpha < \lambda$, $|\{A \cap \alpha \mid A \in S\}| < \mu$. If this holds, then we are done since $\lambda \geq \beth_3(LS(K)) > \beth_2(LS(K))$. If not, replace λ with smallest $\alpha < \lambda$ such that $|\{A \cap \alpha \mid A \in S\}| \geq \mu$. By minimality for all $\beta < \alpha$, $|\{A \cap \beta \mid A \in S\}| < \mu$. Such α might be small, but by the claim $\alpha \geq (\beth_2(LS(K)))^+$, and this is enough for the rest of the argument.

For each $\alpha \leq \lambda$ let $S^0_\alpha := \{\langle A \cap \alpha, \alpha\rangle \mid A \in S\}$. $\bigcup_\alpha S^0_{\alpha<\lambda}$ is a tree when equipped with

$$(A_1, \alpha_1) \leq (A_2, \alpha_2) \iff \alpha_1 \leq \alpha_2 \wedge A_1 = A_2 \cap \alpha_1.$$

Let

$$S^1_\alpha := \{s \in S^0_\alpha \mid |\{t \in S^0_\alpha \mid s \leq t\}| \geq \mu\},$$

and

$$S^1_\lambda := \{s \in S^0_\lambda \mid \forall \alpha < \lambda (s \restriction_\alpha \in S^1_\alpha)\}.$$

We build

1. for $n < \omega$, $S_n \subseteq S^1_\lambda$, and
2. for each $i \in S_n$ and $(A, i) \in S^1_i$, and
   (a) $\lambda > \alpha^A_i(n, 0) > \ldots > \alpha^A_i(n, n-1) > i$, a sequence of ordinals,
   (b) $(D^{(A,i)}_{u,n}, \lambda) \in S^1_\lambda$ for each $u \subseteq n$, and
3. $p_n \in S^{n+2^n}_T(\emptyset)$ for $n < \omega$

such that:

(1) $S_0 = S^1_\lambda$;
(2) $|S_n| \geq (\beth_2(LS(K)))^+$ for all n;
(3) $S_{n+1} \subseteq S_n$ for all n;
(4) The variables of p_n are x_i for $i < n$ ordered naturally and y_S for $S \subseteq n$ ordered by $\subseteq$;
(5) $p_n \subseteq p_{n+1}$ for all n. This means the p_{n+1} restricted to x_i for $i < n$ and y_S for $S \subseteq n$ is equal to p_n;
(6) For all $n < m$, $(A, i) \in S_n$ and $(B, j) \in S_m$, $(A, i) \leq (B, j) \in \bigcup_\alpha S^0_\alpha$
$$\begin{aligned} p_n &= \mathbf{tp}_T(\langle \alpha^A_i(n,0), \ldots \alpha^A_i(n, n-1)\rangle^\frown \langle D^{(A,i)}_{w,n} \mid w \subseteq n\rangle/\emptyset, M) \\ &= \mathbf{tp}_T(\langle \alpha^B_j(m,0), \ldots \alpha^B_j(m, n-1)\rangle^\frown \langle D^{(B,j)}_{w,m} \mid w \subseteq m\rangle/\emptyset, M); \end{aligned}$$
(7) For all $(A, i) \in S_n$ and $w \subseteq n$, $(A, i) \leq (D^{(A,i)}_{w,n}, \lambda)$ and $\alpha^A_i(n, i) \in D^{(A,i)}_{w,n} \iff i \in w$.

Construction: We build these objects by induction on n. When $n = 0$ let $D^{(\emptyset,0)}_{\emptyset,0}$ be any element in S^1_λ. Assume we have built S_n, $\alpha^A_i(n, j)$ for $(A, i) \in S_n$ and p_n.

Fix $s = (A, i) \in S_n$. Clearly $T_s := \{t \in \bigcup_{\beta<\lambda} S^1_\beta \mid s \leq t\}$ is a tree. For every $s \leq t \in S^1_\lambda$, $B_t := \{t^* \mid s \leq t^* \leq t\}$ is a branch of T_s, and $t_1 \neq t_2 \implies B_{t_1} \neq B_{t_2}$. Since
$$|S^0_\lambda - S^1_\lambda| = |\bigcup_{\alpha<\lambda, s\in S^0_\alpha - S^1_\alpha} \{t \in S^0_\lambda \mid s \leq t\}| < \mu,$$
T_s has $\geq \mu$-many branches, and hence $|T_s| > \lambda$. Then for some i', $|T_s \cap S^1_{i'}| > \lambda$. Let $s_j = (A_j, i') \in T_s \cap S^1_{i'}$ for $j < \lambda^+$. Since there are $\leq \lambda$ finite tuples of ordinals $< \lambda$, we may assume $\alpha^{A_j}_{i'}$ are the same for all j. Now let $\alpha^A_i(n+1, k) := \alpha^{A_j}_{i'}(n, k)$ for all $k < n$. Let $\alpha^A_i(n+1, n)$ be the least α such that $s_0(\alpha) \neq s_1(\alpha)$, i.e. $\alpha \in A_0 - A_1$ or $\alpha \in A_1 - A_0$. Without loss of generality assume the latter case. For $w \subseteq (n + 1)$, let $D^{(A,i)}_{w,n+1} := D^{(A_0,i')}_{w,n}$ if $n \notin w$ and $D^{(A,i)}_{w,n+1} := D^{(A_1,i')}_{w,n}$ if $n \in w$.

Note that $i < \alpha^A_i(n + 1, n) < i' < \alpha^A_i(n + 1, n - 1) < \ldots < \alpha_i(n + 1, 0)$. Since $|S_n| =\geq (\beth_2(LS(K)))^+$, and there are $\leq \beth_2(LS(K))$ T-types, by the pigeonhole principle there is $S_{n+1} \subseteq S_n$, $|S_{n+1}| \geq (\beth_2(LS(K)))^+$ such that for all (A, i), $(B, j) \in S_{n+1}$,
$$\mathbf{tp}_T(\langle \alpha^A_i(n, 0), \ldots \alpha^A_i(n + 1, n)\rangle^\frown \langle D^{(A,i)}_{w,n+1} \mid w \subseteq n + 1\rangle/\emptyset, M)$$
is the same, and define this type to be p_{n+1}. This finishes the construction. Note that here since $D^{(A,i)}_{w,n+1}$ is an element of $S^1_\lambda \subseteq S^0_\lambda = S$, i.e. a ϕ-type, the "T-type" of $D^{(A,i)}_{w,n+1}$ is just the T-type of a realization of it, which can be fixed at the beginning of the proof.
$$T^* := T \cup \{\phi(c_i, d_w)^{i\in w}) \mid w \subseteq \omega\} \cup \{p_n(\langle c_i \mid i < n\rangle^\frown \langle d_w \mid w \subseteq \omega\rangle) \mid n < \omega\}$$

is consistent, and by Morley's method we are done. $\square$

Theorem 4.11. If K can encode subsets of $\mu := \beth_{(2^{LS(K)})^+}$, then it can encode subsets of any cardinal. That is, if there are $M \in K$, $\{a_i \mid i < \mu\} \subseteq |M|$, $\{b_w \mid w \subseteq \mu\} \subseteq |M|$ such that for all $w \subseteq \mu$,

$$i \in w \iff \phi(a_i, b_w),$$

then we can replace μ above by any cardinal.

Proof. We fix $\hat{K}$ and ϕ as in the proof of the previous theorem. Let $\lambda = (2^{LS(K)})^+$. Suppose K can encode subsets of $\mu := \beth_{(2^{LS(K)})^+}$. That is, there are $M \in K$, $\{a_i \mid i < \mu\} \subseteq |M|$, $\{b_w \mid w \subseteq \mu\} \subseteq |M|$ such that for all $w \subseteq \mu$,

$$i \in w \iff \phi(a_i, b_w).$$

For each $i_0 < \ldots < i_{n-1} < \mu$ and $u \subseteq n$, choose some subset $w \subseteq \mu$ such that $i_j \in w \iff \phi(a_{i_j}, b_w) \iff j \in u$, and let $b^{i_0,\ldots,i_{n-1}}_{u,n} := b_w$. We build $\langle F_n \subseteq \mu \mid n < \omega\rangle$, $\langle X_{\xi,n} \subseteq \mu \mid \xi \in F_n, n < \omega\rangle$ and $p_n \in S^{n+2^n}_T(\emptyset)$ such that:

(1) For all $n < \omega$, $|F_n| = \lambda$;
(2) $|X_{\xi,n}| > \beth_\beta(2^{LS(K)})$ when ξ is the β-th element of F_n;
(3) $p_n(\langle a_{i_j} \mid j < n\rangle^\frown\langle b^{i_0,\ldots,i_{n-1}}_{u,n} \mid u \subseteq n\rangle)$.

Let $F_0 = \lambda$ and $X_{\xi,0} := \mu$ for all ξ. Suppose we have constructed everything for stage n. Fix $g : \lambda \to F_n$ an increasing enumeration. Let $G_n := \{g(\beta + n + 1) \mid \beta < \lambda\}$. For each $\xi = g(\beta + n + 1) \in G_n$, consider the map $\langle i_j \mid j < n\rangle \mapsto \mathbf{tp}_T(\langle a_{i_j} \mid j < n + 1\rangle^\frown\langle b^{i_0,\ldots,i_n}_{u,n+1} \mid u \subseteq n + 1\rangle/\emptyset, M)$ from $[X_{\xi,n}]^{n+1}$ (increasing $(n + 1)$-tuples from $X_{\xi,n}$) to $S^{n+2^n}_T(\emptyset)$. Since $|X_{\xi,n}| > \beth_{\beta+n+1}((2^{LS(K)})^+)$, by the Erdős-Rado theorem, there is a monochromatic subset $X_{\xi,n+1} \subseteq X_{\xi,n}$ such that $|X_{\xi,n+1}| > \beth_\beta((2^{LS(K)})^+)$. I.e. there is a type $p_{\xi,n+1}$ such that for all $i_0 < \ldots < i_n$, $\mathbf{tp}_T(\langle a_{i_j} \mid j < n\rangle^\frown\langle b^{i_0,\ldots,i_n}_{u,n+1} \mid u \subseteq n + 1\rangle/\emptyset, M) = p_{\xi,n+1}$. By the pigeonhole principle there is $F_{n+1} \subseteq G_n$ of cardinality λ and p_{n+1} such that for all $\xi \in F_{n+1}$, $p_{\xi,n+1} = p_{n+1}$.

Then

$$T^* := T \cup \{\phi(c_i, d_w)^{i\in w}) \mid w \subseteq \kappa\} \cup \{p_n(\langle c_{i_j} \mid j < n\rangle^\frown\langle d_w \mid w \subseteq w\rangle) \mid n < \omega, i_0 < \ldots < i_{n-1} < \kappa\}$$

is consistent for any cardinal κ. By Morley's method we are done. $\square$

Lemma 4.12 (Morley's method)**.** Let T be a first order theory with built-in Skolem functions and Γ a set of T-types. Let $\langle c_i \mid i < \alpha\rangle$ be new constants. Let p_S be a T-type in $|S|$ variables for every finite subset S of α, and T^* a theory not containing any of the new constants such that:

(1) $T^* \supseteq T \cup \{p_S(\langle c_\gamma \mid \gamma \in S\rangle) \mid S \subseteq \alpha \text{ finite}\}$ is consistent;
(2) Each p_S is realized in some $M \in EC(T, \Gamma)$.

Then there is $N \in EC(T^*, \Gamma)$.

Proof. Let M be a model of T^* and without loss of generality $M = EM(\{c_i \mid i < \alpha\})$. We show that M omits all types from Γ. Suppose not, i.e. $a \in |M|$ realizes some $q \in \Gamma$. Write a as $\tau^M(c_{i_0}, \dots, c_{i_k})$ for some term τ in the language of T. Let $S := \{c^M_{i_0}, \dots, c^M_{i_k}\}$ and $\langle b_0, \dots, b_k \rangle \subseteq N^* \in EC(T, \Gamma)$ realizing p_S. Then for some $\varphi(y) \in q$, $N^* \models \neg\varphi(\tau(b_0, \dots, b_k))$. As p_S is complete, $\neg\varphi(\tau(x_0, \dots, x_k)) \in p_S$. Thus $M \models \not\varphi(\tau(c_{i_0}, \dots, c_{i_k}))$, i.e. $M \models \neg\varphi(a)$, so a does not realize q. $\square$

References

[Bal09] J.T. Baldwin, *Categoricity*, University lecture series, American Mathematical Society, 2009.

[BET07] John T. Baldwin, Paul C. Eklof, and Jan Trlifaj, *$N^\perp$ as an abstract elementary class*, Annals of Pure and Applied Logic **149** (2007), no. 1, 25–39.

[BG17] Will Boney and Rami Grossberg, *Forking in short and tame abstract elementary classes*, Annals of Pure and Applied Logic **168** (2017), no. 8, 1517–1551.

[BGKV16] Will Boney, Rami Grossberg, Alexei Kolesnikov, and Sebastien Vasey, *Canonical forking in aecs*, Annals of Pure and Applied Logic **167** (2016), no. 7, 590–613.

[DS78] Keith J. Devlin and Saharon Shelah, *A weak version of $\diamondsuit$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$*, Israel J. Math. **29** (1978), no. 2-3, 239–247, Paper number 65 in Shelah's publication list. MR 0469756

[Gro02] Rami Grossberg, *Classification theory for abstract elementary classes*, In Logic and Algebra, Yi Zhang editor, Contemporary Mathematics 302, AMS,(2002), 165–203, 2002.

[GS] Rami P. Grossberg and Saharon Shelah, *On Hanf numbers of the infinitary order property*, Paper number 259 in Shelah's publication list.

[GS86] Rami Grossberg and Saharon Shelah, *A nonstructure theorem for an infinitary theory which has the unsuperstability property*, Illinois J. Math. **30** (1986), no. 2, 364–390, Paper number 238 in Shelah's publication list. MR 840135

[GV06] Rami Grossberg and Monica VanDieren, *Categoricity from one successor cardinal in tame abstract elementary classes*, Journal of Mathematical Logic **6** (2006), no. 02, 181–201.

[GV17] Rami Grossberg and Sebastien Vasey, *Equivalent definitions of superstability in tame abstract elementary classes*, The Journal of Symbolic Logic **82** (2017), no. 4, 1387–1408.

[JS13] Adi Jarden and Saharon Shelah, *Non-forking frames in abstract elementary classes*, Ann. Pure Appl. Logic **164** (2013), no. 3, 135–191, Paper number 875 in Shelah's publication list. MR 3001542

[Kei76] H. Jerome Keisler, *Six classes of theories*, Journal of the Australian Mathematical Society **21** (1976), no. 3, 257–266.

[Les05] Olivier Lessmann, *Upward categoricity from a successor cardinal for tame abstract classes with amalgamation*, The Journal of Symbolic Logic **70** (2005), no. 2, 639–660.

[Leu23] Samson Leung, *Hanf number of the first stability cardinal in aecs*, Annals of Pure and Applied Logic **174** (2023), no. 2, 103201.

[MA20] Marcos Mazari-Armida, *Non-forking w-good frames*, Archive for Mathematical Logic **59** (2020), no. 1, 31–56.

[MAR23] Marcos Mazari-Armida and Jiri Rosicky, *Relative injective modules, superstability and noetherian categories*, 2023.

[She71a] Saharon Shelah, *Stability, the f.c.p., and superstability; model theoretic properties of formulas in first order theory*, Ann. Math. Logic **3** (1971), no. 3, 271–362, Paper number 10 in Shelah's publication list. MR 0317926

[She71b] ______, *The number of non-isomorphic models of an unstable first-order theory* , Israel J. Math. **9** (1971), 473–487, Paper number 12 in Shelah's publication list. MR 0278926

[She75] ______, *Categoricity in $\aleph_1$ of sentences in $L_{\omega_1,\omega}(Q)$*, Israel J. Math. **20** (1975), no. 2, 127–148, Paper number 48 in Shelah's publication list. MR 0379177

[She78] Saharon Shelah, *Classification theory and the number of non-isomorphic models*, Studies in logic and the foundations of mathematics, vol. 92, North-Holland, 1978.

[She87] Saharon Shelah, *Classification of nonelementary classes. II. Abstract elementary classes*, Classification theory (Chicago, IL, 1985), Lecture Notes in Math., vol. 1292, Springer, Berlin, 1987, Paper number 88 in Shelah's publication list, pp. 419–497. MR 1033034

[She99] ______, *Categoricity for abstract classes with amalgamation*, Ann. Pure Appl. Logic **98** (1999), no. 1-3, 261–294, Paper number 394 in Shelah's publication list. MR 1696853

[She01] ______, *Categoricity of an abstract elementary class in two successive cardinals*, Israel J. Math. **126** (2001), 29–128, Paper number 576 in Shelah's publication list. MR 1882033

[She09a] ______, *Categoricity in abstract elementary classes: going up inductively*, 2009, Paper number 600 in Shelah's publication list, Ch. II of [Sh:h].

[She09b] ______, *Categoricity of an abstract elementary class in two successive cardinals, revisited*, 2009, Ch. 6 of [Sh:i].

[She09c] Saharon Shelah, *Classification theory for abstract elementary classes*, Studies in Logic: Mathematical logic and foundations, vol. 18, College Publications, 2009.

[She09d] ______, *Classification theory for abstract elementary classes 2*, Studies in Logic: Mathematical logic and foundations, vol. 20, College Publications, 2009.

[Vas16a] Sebastien Vasey, *Building independence relations in abstract elementary classes*, Annals of Pure and Applied Logic **167** (2016), no. 11, 1029–1092.

[Vas16b] ______, *Forking and superstability in tame aecs*, The Journal of Symbolic Logic **81** (2016), no. 1, 357–383.

[Vas16c] ______, *Infinitary stability theory*, Archive for Mathematical Logic **55** (2016), 567–592.

[Zil05] Boris Zilber, *A categoricity theorem for quasi-minimal excellent classes*, Logic and its Applications (Andreas Blass and Yi Zhang, ed.), **380** (2005), 297–306.

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