BUILDING MODELS IN SMALL CARDINALS IN LOCAL ABSTRACT ELEMENTARY CLASSES

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Abstract. There are many results in the literature where superstablity-like independence notions, without any categoricity assumptions, have been used to show the existence of larger models. In this paper we show that *stability* is enough to construct larger models for small cardinals assuming a mild locality condition for Galois types.

Theorem 0.1. Suppose $\lambda < 2^{\aleph_0}$. Let $\mathbf{K}$ be an abstract elementary class with $\lambda \geq \mathrm{LS}(\mathbf{K})$. Assume $\mathbf{K}$ has amalgamation in λ, no maximal model in λ, and is stable in λ. If $\mathbf{K}$ is $(< \lambda^+, \lambda)$-local, then $\mathbf{K}$ has a model of cardinality λ^{++}.

The set theoretic assumption that $\lambda < 2^{\aleph_0}$ and model theoretic assumption of stability in λ can be weaken to the model theoretic assumptions that $|\mathbf{S}^{na}(M)| < 2^{\aleph_0}$ for every $M \in \mathbf{K}_\lambda$ and stability for λ-algebraic types in λ.

We further use the result just mentioned to provide a positive answer to Grossberg's question for small cardinals assuming a mild locality condition for Galois types and without any stability assumptions. This last result relies on an unproven claim of Shelah [Sh:h, VI.2.11.(2)] (Fact 4.5 of this paper) which we were unable to verify.

1. Introduction

Abstract elementary classes (AECs for short) were introduced by Shelah [Sh88] to study classes of structures axiomatized in several infinitary logics. Given an abstract elementary class $\mathbf{K}$ and λ an infinite cardinal, $\mathbb{I}(\mathbf{K}, \lambda)$ denotes the number of non-isomorphic models in $\mathbf{K}$ of cardinality λ. One of the main test questions in the development of abstract elementary classes is Grossberg's question [Sh576, Problem (5), p. 34][1]:

Question 1.1. Let $\mathbf{K}$ be an AEC and $\lambda \geq \mathrm{LS}(\mathbf{K})$ be an infinite cardinal. If $\mathbb{I}(\mathbf{K}, \lambda) = 1$ and $1 \leq \mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$, must $\mathbf{K}$ have a model of cardinality λ^{++}?

The question is still open despite many approximations: [Sh88, 3.7], [Sh576], [Sh:h, §VI.0.(2)], [Sh:h, §II.4.13.3], [JaSh13, 3.1.9], [Vas16, 8.9], [Vas18b, 12.1], [ShVas18, 5.8], [MaVa18, 3.3, 4.4], [Maz20, 4.2], [Vas22, 1.6, 3.7, 5.4], [Leu23, 4.9].

A key intermediate step to answer Grossberg's question has been to show that stability and even the existence of a superstablity-like independence notion follow from categoricity in several cardinals. Recently, there are many results where superstablity-like independence notions, without any categoricity assumptions, have been used to show the existence of larger models [Sh:h, §II.4.13.3], [JaSh13, 3.1.9], [Vas16, 8.9], [Maz20, 4.2]. In this paper, we show that

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[1]An earlier and weaker version of this question (also due to Grossberg) is Question 4 on page 421 of [Sh88].

stability, without any categoricity assumptions, is enough to construct larger models for small cardinals assuming a mild locality condition for Galois types.

Theorem 3.12. Suppose $\lambda < 2^{\aleph_0}$. Let $\mathbf{K}$ be an abstract elementary class with $\lambda \geq \mathrm{LS}(\mathbf{K})$. Assume $\mathbf{K}$ has amalgamation in λ, no maximal model in λ, and is stable in λ. If $\mathbf{K}$ is $(<\lambda^+, \lambda)$-local, then $\mathbf{K}$ has a model of cardinality λ^{++}.

To help us compare our results with previous results, let us recall the following three frameworks: universal classes [Tar54], [Sh300], tame AECs [GrVan06] and local AECs [Sh576] [BaLe06]. The first is a semantic assumption on the AEC while the other two are locality assumptions on Galois types (see Definition 2.4 and Definition 2.5). The relation between these frameworks is as follows: universal classes are $(<\aleph_0, \lambda)$-tame for every $\lambda \geq \mathrm{LS}(\mathbf{K})$ [Vas17, 3.7] and $(<\aleph_0, \lambda)$-tame AECs are $(<\lambda^+, \lambda)$-local for every $\lambda \geq \mathrm{LS}(\mathbf{K})$. The first inclusion is proper and the second inclusion is not known to be proper (see Question 2.8).

When the AEC has a countable Löwenheim-Skolem-Tarski number and $\lambda = \aleph_0$, Theorem 3.12 for $(<\aleph_0, \aleph_0)$-tame AECs can be obtained using [ShVas18, 4.7], [ShVas18, 5.8], [Sh:h, II.4.13][2], but the result has never been stated in the literature. The argument presented in this paper is significantly simpler than the argument using the results of Shelah and Vasey. Moreover, the result is new for $(\aleph_0, \aleph_0)$-local AECs. Furthermore, for $\lambda > \aleph_1$, Theorem 3.12 is even new for universal classes.

Another result similar to Theorem 3.12 is [Vas18b, 12.1]. The main difference is that Vasey's result has the additional assumption that the AEC is categorical in λ. Moreover, Vasey assumes tameness while we only assume the weaker property of locality for Galois types. It is worth mentioning that Vasey does not assume that $\lambda < 2^{\aleph_0}$, but this is a weak assumption as long as λ is a *small* cardinal.

The main difference between the proof of Theorem 3.12 and the previous results is that we focus on finding *one* good type instead of a *family* of good types. A good type in this paper is a λ-unique type (Definition 3.8). Once we have this good type, we carefully build a chain of types above this type to show that every model of cardinality λ^+ has a proper extension and hence show the existence of a model of cardinality λ^{++}.

The set theoretic assumption that $\lambda < 2^{\aleph_0}$ and model theoretic assumption of stability in λ can be weaken to the model theoretic assumptions that $|\mathbf{S}^{na}(M)| < 2^{\aleph_0}$ for every $M \in \mathbf{K}_\lambda$ and stability for λ-algebraic types in λ (see Theorem 3.11). For instance, for an AEC with a countable Löwenheim-Skolem-Tarski number and $\lambda = \aleph_0$, these assumptions are weaker if $\mathbf{K}$ is not stable in $\aleph_0$ and $2^{\aleph_0} > \aleph_1$. The result in this generality is new even for universal classes.

The main result of the second part of the paper is a positive answer to Grossberg's question for small cardinals assuming a mild locality condition for Galois types and without any stability assumptions. This result relies on a result of Shelah [Sh:h, VI.2.11.(2)] (Fact 4.5 of this paper) for which Shelah does not provide an argument, for which the *standard* argument does not seem to work, and which we were unable to verify. See Remark 4.6 for more details.

Theorem 4.11. Let λ be an infinite cardinal such that $2^\lambda < 2^{\lambda^+}$ and $\lambda^+ < 2^{\aleph_0}$. Let $\mathbf{K}$ be an abstract elementary class with $\lambda \geq \mathrm{LS}(\mathbf{K})$ and suppose Fact 4.5 holds. Assume $\mathbb{I}(\mathbf{K}, \lambda) = 1 \leq \mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$. If $\mathbf{K}$ is $(<\lambda^+, \lambda)$-local, then $\mathbf{K}$ has a model of cardinality λ^{++}.

[2] We were unaware of this argument until Sebastien Vasey pointed it out when we showed him a final draft of the paper.

Theorem 4.11 follows directly from the strengthening of Theorem 3.12 mentioned two paragraph above (Theorem 3.11) once stability for λ-algebraic types in λ has been determined. It is in the proof of stability for λ-algebraic types in λ where we use the unproven claim of Shelah (Fact 4.5).

The only results toward an answer to Grossberg's question where the number of non-isomorphic models are *only* bounded in *two* cardinals without any stability or superstability like assumptions are [Sh88, 3.7], [MaVa18, 3.3, 4.4] and [Vas22, 1.6]. All of these results assume that the AEC has a countable Löwenheim-Skolem-Tarski number, and that the cardinals under consideration are $\aleph_0$ and $\aleph_1$. Shelah's result assumes that the AEC is *definable*[3] while the other results assume that $\mathbf{K}$ is *close* to a universal class.

Compared to the results mentioned above, we do not assume that the AEC has a countable Löwenheim-Skolem-Tarski number or that $\lambda = \aleph_0$. When the AEC has a countable Löwenheim-Skolem-Tarski number and $\lambda = \aleph_0$, Theorem 4.11 is known for universal classes [MaVa18, 3.3] but it is new for $(< \aleph_0, \aleph_0)$-tame AECs. Moreover, for $\lambda > \aleph_1$, Theorem 4.11 is new even for universal classes. Both [MaVa18, 3.3, 4.4] and [Vas22, 1.6] assume that $2^{\aleph_0} < 2^{\aleph_1}$, but the assumption that $\lambda^+ < 2^{\aleph_0}$ is new although this is a weak assumption as long as λ is a *small* cardinal.

The paper is organized as follows. Section 2 presents necessary background. Section 3 shows how to construct larger models from stability. Section 4 presents an answer to Grossberg's question for small cardinals assuming a mild locality condition for Galois types and without any stability assumptions, but using an unproven claim of Shelah.

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2. Preliminaries

We assume the reader has some familiarity with abstract elementary classes as presented in [Bal09, §4 - 8], [Gro02] or [Sh:h, §2], but we recall the main notions used in this paper.

An AEC is a pair $\mathbf{K} = (K \leq_{\mathbf{K}})$ where K is a class of structures in a fixed language and $\leq_{\mathbf{K}}$ is a partial order on K extending the substructure relation such that $\mathbf{K}$ is closed under isomorphisms and satisfies the coherence property, the Löwenheim-Skolem-Tarski axiom and the Tarski-Vaught axioms. The reader can consult the definition in [Bal09, 4.1].

Notation 2.1. For any structure M, we denote its universe by $|M|$, and its cardinality by $\|M\|$. For a cardinal λ, we let $\mathbf{K}_\lambda = \{M \in \mathbf{K} : \|M\| = \lambda\}$. When we write $M \leq_{\mathbf{K}} N$ we assume that $M, N \in \mathbf{K}$.

For an AEC $\mathbf{K}$, $\mathbf{K}$ has the amalgamation property if for every $M_0 \leq_{\mathbf{K}} M_l$ for $\ell = 1, 2$, there is $N \in \mathbf{K}$ and $\mathbf{K}$-embeddings $f_\ell : M_\ell \to N$ for $\ell = 1, 2$ such that $f_1 \restriction_{M_0} = f_2 \restriction_{M_0}$; $\mathbf{K}$ has the joint embedding property if for every M_0, $M_1 \in \mathbf{K}$ there is $N \in \mathbf{K}$ such that M_0, M_1 $\mathbf{K}$-embed

[3]More precisely a $\mathrm{PC}_{\aleph_0}$ class

into N; and $\mathbf{K}$ has no maximal models if every $M \in \mathbf{K}$ has a proper $\leq_{\mathbf{K}}$-extension in $\mathbf{K}$. For a property P, we say that $\mathbf{K}$ has P in λ if $\mathbf{K}_\lambda$ has the property P.

For an AEC $\mathbf{K}$ and $\lambda \geq \mathrm{LS}(\mathbf{K})$, we denote by $\mathbb{I}(\mathbf{K}, \lambda)$ the number of non-isomorphic models in $\mathbf{K}_\lambda$. If $\mathbb{I}(\mathbf{K}, \lambda) = 1$, we say that $\mathbf{K}$ is λ-categorical.

Throughout the rest of this section $\mathbf{K}$ is always an abstract elementary class and λ is always a cardinal greater than or equal to the Löwenheim-Skolem-Tarski number of $\mathbf{K}$.

Fact 2.2 ([Sh88, 3.5], [Gro02, 4.3])**.** $(2^\lambda < 2^{\lambda^+})$ If $\mathbb{I}(\mathbf{K}, \lambda) = 1 \leq \mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$, then $\mathbf{K}$ has amalgamation in λ.

We recall the notion of a Galois type. These were originally introduced by Shelah.

Definition 2.3.

(1) For $(b_1, A_1, N_1), (b_2, A_2, N_2)$ such that $N_\ell \in \mathbf{K}$, $A_\ell \subseteq |N|$ and $b_\ell \in N_\ell$ for $\ell = 1, 2$, $(b_1, A_1, N_1) E_{\mathrm{at}} (b_2, A_2, N_2)$ if $A := A_1 = A_2$, and there exist $\mathbf{K}$-embeddings $f_\ell : N_\ell \to N$ for $\ell = 1, 2$ such that $f_1(b_1) = f_2(b_2)$ and $f_1 \restriction_A = f_2 \restriction_A$. Let E be the transitive closure of E_{at}.
(2) Given (b, A, N), where $N \in \mathbf{K}$, $A \subseteq |N|$, and $b \in N$, the *Galois type of* b *over* A *in* N, denoted by $\mathbf{gtp}(b/A, N)$, is the equivalence class of (b, A, N) modulo E.
(3) For $M \in \mathbf{K}$, $\mathbf{S}(M) := \{\mathbf{gtp}(a/M, N) : M \leq_{\mathbf{K}} N \text{ and } a \in |N|\}$ denotes the set of all Galois types over M and $\mathbf{S}^{na}(M) := \{\mathbf{gtp}(a/M, N) : M \leq_{\mathbf{K}} N \text{ and } a \in |N| \backslash |M|\}$ denotes the set of all *non-algebraic types* over M.
(4) Given $p = \mathbf{gtp}(b/A, N)$ and $C \subseteq A$, let $p \restriction C = [(b, C, N)]_E$. Given $M \leq_{\mathbf{K}} N$, $p \in \mathbf{S}(N)$ and $q \in \mathbf{S}(M)$, p *extends* q, denoted by $q \leq p$, if $p \restriction_M = q$.

Since Galois types are equivalence classes, they might not have a local behaviour. The two following notions were isolated as possible instances of local behaviour. Tameness appears in some of the arguments of [Sh394] and was isolated in [GrVan06]. Locality appears for the first time in-print in [Sh576].

Definition 2.4.

(1) $\mathbf{K}$ is (κ, λ)*-tame* if for every $M \in \mathbf{K}_\lambda$ and every $p, q \in \mathbf{S}(M)$, if $p \neq q$, then there is $A \subseteq |M|$ of cardinality κ such that $p \restriction_A \neq q \restriction_A$.
(2) $\mathbf{K}$ is $(< \kappa, \lambda)$*-tame* if for every $M \in \mathbf{K}_\lambda$ and every $p, q \in \mathbf{S}(M)$, if $p \neq q$, then there is $A \subseteq |M|$ of cardinality less than κ such that $p \restriction_A \neq q \restriction_A$.

Definition 2.5.

(1) $\mathbf{K}$ is (κ, λ)*-local* if for every $M \in \mathbf{K}_\lambda$, every increasing continuous chain $\langle M_i : i < \kappa \rangle$ such that $M = \bigcup_{i<\kappa} M_i$ and every $p, q \in \mathbf{S}(M)$, if $p \restriction_{M_i} = q \restriction_{M_i}$ for all $i < \kappa$ then $p = q$.
(2) $\mathbf{K}$ is $(< \kappa, \lambda)$*-local* if $\mathbf{K}$ is (μ, λ)-local for all $\mu < \kappa$.

Below are some relations between tameness and locality.

Proposition 2.6. Let $\lambda \geq \mathrm{LS}(\mathbf{K})$.

(1) If $\mathbf{K}$ is $(< \aleph_0, \lambda)$-tame, then $\mathbf{K}$ is $(< \lambda^+, \lambda)$-local.
(2) Assume $\lambda > \mathrm{LS}(\mathbf{K})$. If $\mathbf{K}$ is (λ, λ)-local, then $\mathbf{K}$ is $(< \lambda, \lambda)$-tame.
(3) If $\mathbf{K}$ is (μ, μ)-local for every $\mu \leq \lambda$, then $\mathbf{K}$ is $(\mathrm{LS}(\mathbf{K}), \mu)$-tame for every $\mu \leq \lambda$.
(4) Assume $\lambda \geq \kappa$, cf $(\kappa) > \chi$. If $\mathbf{K}$ is (χ, λ)-tame, then $\mathbf{K}$ is (κ, λ)-local.

Proof.

(1) Straightforward.
(2) Let $M \in \mathbf{K}_\lambda$ and $p, q \in \mathbf{S}(M)$ such that $p \restriction_A = q \restriction_A$ for every $A \subseteq |M|$ with $|A| < \lambda$. Let $\langle M_i : i < \lambda \rangle$ be an increasing continuous chain such that $M = \bigcup_{i<\lambda} M_i$ and $\|M_i\| \leq \mathrm{LS}(\mathbf{K}) + |i|$ for every $i < \lambda$. Since $\|M_i\| < \lambda$ for every $i < \lambda$, $p \restriction_{M_i} = q \restriction_{M_i}$ for every $i < \lambda$. Therefore, $p = q$ as $\mathbf{K}$ is (λ, λ)-local.
(3) Similar to (2), see also [BaLe06, 1.18].
(4) This is [BaSh08, 1.11]

$\square$

Remark 2.7. Universal classes are $(< \aleph_0, \lambda)$-tame for every $\lambda \geq \mathrm{LS}(\mathbf{K})$ [Vas17, 3.7], Quasiminimal AECs are $(< \aleph_0, \lambda)$-tame for every $\lambda \geq \mathrm{LS}(\mathbf{K})$ [Vas18a, 4.18] and many natural AECs of module are $(< \aleph_0, \lambda)$-tame for every $\lambda \geq \mathrm{LS}(\mathbf{K})$ (see for example [Maz23, §3]). The main results of the paper assume that the AEC is $(< \lambda^+, \lambda)$-local, so they apply to all of these classes.

On the other hand there are AECs which are not $(\aleph_1, \aleph_1)$-local [BaSh08] and which are not tame [BaKo09].

Despite the importance of tameness in the development of AECs, the following question is still open.

Question 2.8. If $\mathbf{K}$ is $(\aleph_0, \aleph_0)$-local, is $\mathbf{K}$ $(< \aleph_0, \aleph_0)$-tame?

Since we will be building chains of types the following will be important.

Definition 2.9. Let $\langle M_i : i < \delta \rangle$ be an increasing continuous chain. A sequence of types $\langle p_i \in \mathbf{S}(M_i) : i < \delta \rangle$ is *coherent* if there are (a_i, N_i) for $i < \delta$ and $f_{j,i} : N_j \to N_i$ for $j < i < \delta$ such that:

(1) $f_{k,i} = f_{j,i} \circ f_{k,j}$ for all $k < j < i$.
(2) $\mathbf{gtp}(a_i/M_i, N_i) = p_i$.
(3) $f_{j,i} \restriction_{M_j} = id_{M_j}$.
(4) $f_{j,i}(a_j) = a_i$.

Recall that if $\langle M_i : i < \omega \rangle$ is an increasing chain, then any increasing sequence of types $\langle p_i \in \mathbf{S}(M_i) : i < \omega \rangle$ is coherent. The following fact is well-known, see for example [Maz20, 3.14].

Fact 2.10. Let δ be a limit ordinal and $\langle M_i : i \leq \delta \rangle$ be an increasing continuous chain. If $\langle p_i \in \mathbf{S}^{na}(M_i) : i < \delta \rangle$ is a coherent sequence of types, then there is $p \in \mathbf{S}^{na}(M_\delta)$ such that $p \geq p_i$ for every $i < \delta$ and $\langle p_i \in \mathbf{S}^{na}(M_i) : i < \delta + 1 \rangle$ is coherent.

3. Main results

In this section we prove the main results of the paper. Throughout this section $\mathbf{K}$ is always an abstract elementary class and λ is always a cardinal greater than or equal to the Löwenheim-Skolem-Tarski number of $\mathbf{K}$. We begin by recalling the following notions that appear in [Yan] and [Sh:h, §VI].

Definition 3.1.

- $p = \mathbf{gtp}(a/M, N)$ has the λ-*extension property* if for every $M \leq_{\mathbf{K}} M' \in \mathbf{K}_\lambda$, there is $q \in \mathbf{S}^{na}(M')$ extending p. In this case we say $p \in \mathbf{S}^{\lambda-ext}(M)$.[4]

[4] These types are also called *big types* in the literature, see for example [Sh48] and [Les05].

- $p = \mathbf{gtp}(a/M, N)$ is λ-*algebraic* if $p \in \mathbf{S}^{na}(M) - \mathbf{S}^{\lambda-ext}(M)$. Let $\mathbf{S}^{\lambda-al}(M)$ denote the λ-algebraic types over M.

Observe that if p has the λ-extension property then p is non-algebraic.

Fact 3.2 ([Sh:h, VI.2.5(2B)])**.** Assume $\mathbf{K}$ has amalgamation in λ and no maximal model in λ. $\mathbf{gtp}(a/M, N)$ has $\geq \lambda^+$ realizations in some $M' \in \mathbf{K}$ such that $M \leq_{\mathbf{K}} M'$ if and only if $\mathbf{gtp}(a/M, N)$ has the λ-extension property.

Recall that an AEC $\mathbf{K}$ is *stable in* λ if $|\mathbf{S}(M)| \leq \lambda$ for every $M \in \mathbf{K}_\lambda$. We introduce a weakening of stability.

Definition 3.3. $\mathbf{K}$ is stable for λ-algebraic types in λ if for all $M \in \mathbf{K}_\lambda$, $|\mathbf{S}^{\lambda-al}(M)| \leq \lambda$.

Remark 3.4. Stability for λ-algebraic types in λ is strictly weaker than stability in λ. Consider the case where $\mathbf{K}$ is an elementary class which is unstable in λ. In that case, all non-algebraic types have the extension property. Thus $\mathbf{S}^{\lambda-al}(M) = \emptyset$ for all $M \in \mathbf{K}_\lambda$, but for some $M \in \mathbf{K}_\lambda$, $|\mathbf{S}(M)| > \lambda$.

We show that there are types with the λ-extension property.

Lemma 3.5. Assume that $\mathbf{K}$ has amalgamation in λ and no maximal model in λ. If $\mathbf{K}$ is stable for λ-algebraic types in λ, then there is $p \in \mathbf{S}^{\lambda-ext}(M)$ for every $M \in \mathbf{K}_\lambda$.

Proof. Fix $M \in \mathbf{K}_\lambda$. There are two cases to consider:

Case 1: $|\mathbf{S}^{na}(M)| \geq \lambda^+$. This follows directly from the assumption that $\mathbf{K}$ is stable for λ-algebraic types in λ.

Case 2: $|\mathbf{S}^{na}(M)| \leq \lambda$. Since $\mathbf{K}$ has no maximal model in λ, there is $M \leq_{\mathbf{K}} N \in \mathbf{K}_{\lambda^+}$. Let $\Phi : |N| \backslash |M| \to \mathbf{S}^{na}(M)$ be given by $a \mapsto \mathbf{gtp}(a/M, N)$. Since $\| N| \backslash |M \| = \lambda^+$ and $|\mathbf{S}^{na}(M)| \leq \lambda$, by the pigeonhole principle there is $q \in \mathbf{S}^{na}(M)$ such that $|\{a \in |N| \backslash |M| : \Phi(a) = q\}| \geq \lambda^+$. That is, q has λ^+-many realizations in N. Hence q has the the λ-extension property by Fact 3.2. $\square$

We will use the following strengthening of the extension property.

Definition 3.6. $p = \mathbf{gtp}(a/M, N)$ has the λ-*strong extension property* if for every $M \leq_{\mathbf{K}} M' \in \mathbf{K}_\lambda$, there is $q \in \mathbf{S}^{\lambda-ext}(M')$.

We show that the strong extension property is the same as the extension property if $\mathbf{K}$ is stable for λ-algebraic types.

Lemma 3.7. Assume that $\mathbf{K}$ has amalgamation in λ, no maximal model in λ and is stable for λ-algebraic types in λ. Let $M \in \mathbf{K}_\lambda$ and $p \in \mathbf{S}^{na}(M)$, $p \in \mathbf{S}^{\lambda-ext}(M)$ if and only p has the λ-strong extension property.

Proof. We only need to show the forward direction. Let $N \geq_{\mathbf{K}} M$ and $\{a_i \in |N| : i < \lambda^+\}$ realizing p. Let $M \leq_{\mathbf{K}} M^* \in \mathbf{K}_\lambda$. Using amalgamation in λ we may assume that $M^* \leq_{\mathbf{K}} N$. Moreover, we may assume without loss of generality that for all $i < \lambda^+$, $a_i \notin |M^*|$. If not, subtract those a_i that are in M^*. Observe that $\mathbf{gtp}(a_i/M^*, N) \geq p$ for all $i < \lambda^+$. If $|\{\mathbf{gtp}(a_i/M^*, N) : i < \lambda^+\}| = \lambda^+$, we are done by stability for λ-algebraic types. Otherwise $|\{\mathbf{gtp}(a_i/M^*, N) : i < \lambda^+\}| \leq \lambda$. Then a similar argument to that of Case 2 of the previous lemma can be used to obtain result. $\square$

Recall the following notion. This notion was first introduced by Shelah in [Sh48, 6.1], called minimal types there. Note that this is a different notion from the minimal types of [Sh576]. These types are also called *quasiminimal types* in the literature, see for example [Zil05] and [Les05].

Definition 3.8. $p = \mathbf{gtp}(a/M, N)$ is a λ-*unique type* if

(1) $p = \mathbf{gtp}(a/M, N)$ has the λ-extension property.
(2) For every $M \leq_{\mathbf{K}} M' \in \mathbf{K}_\lambda$, p has at most one extension $q \in \mathbf{S}^{\lambda-ext}(M')$.

In this case we say that $p \in \mathbf{S}^{\lambda-unq}(M)$.

We show the existence of λ-unique types.

Lemma 3.9. Assume that $\mathbf{K}$ has amalgamation in λ, no maximal model in λ and is stable for λ-algebraic types in λ. If $|\mathbf{S}^{na}(M)| < 2^{\aleph_0}$ for every $M \in \mathbf{K}_\lambda$, then for every $M_0 \in \mathbf{K}_\lambda$ and $p \in \mathbf{S}^{\lambda-ext}(M_0)$, there is $M_1 \in \mathbf{K}_\lambda$ and $q \in \mathbf{S}^{\lambda-unq}(M_1)$ such that $M_0 \leq_{\mathbf{K}} M_1$ and q extends p.

Proof. Assume for the sake of contradiction that this is not the case. Let $M_0 \in \mathbf{K}_\lambda$ and $p \in \mathbf{S}^{\lambda-ext}(M_0)$ without a λ-unique type above it.

We build $\langle M_n : n < \omega\rangle$ and $\langle p_\eta : \eta \in 2^{<\omega}\rangle$ by induction such that:

(1) $p_{\langle\rangle} = p$;
(2) for every $\eta \in 2^{<\omega}$, $p_\eta \in \mathbf{S}^{\lambda-ext}(M_{\ell(\eta)})$;
(3) for every $\eta \in 2^{<\omega}$, $p_{\eta^\frown 0} \neq p_{\eta^\frown 1}$.

$\boxed{\text{Construction}}$ The base step is given so we do the induction step. By induction hypothesis we have $\langle p_\eta \in \mathbf{S}^{\lambda-ext}(M_n) : \eta \in 2^n\rangle$. Since there is no λ-unique type above $p_{\langle\rangle}$ and by Lemma 3.7, for every $\eta \in 2^n$ there are $N_\eta \in \mathbf{K}_\lambda$ and $q^0_\eta, q^1_\eta \in \mathbf{S}^{\lambda-ext}(N_\eta)$ such that $q^0_\eta, q^1_\eta \geq p_\eta$ and $q^0_\eta \neq q^1_\eta$. Using amalgamation in λ we build $M_{n+1} \in \mathbf{K}_\lambda$ and $\langle f_\eta : N_\eta \underset{M_n}{\longrightarrow} M_{n+1} : \eta \in 2^n\rangle$. Now for every $\eta \in 2^n$, let $p_{\eta^\frown 0}, p_{\eta^\frown 1} \in \mathbf{S}^{\lambda-ext}(M_{n+1})$ such that $p_{\eta^\frown 0} \geq f_\eta(q^0_\eta)$ and $p_{\eta^\frown 1} \geq f_\eta(q^1_\eta)$. These exist by Lemma 3.7. It is easy to show that M_{n+1} and $\langle p_{\eta^\frown \ell} : \eta \in 2^n, \ell \in \{0, 1\}\rangle$ are as required.

$\boxed{\text{Enough}}$ Let $N := \bigcup_{n<\omega} M_n \in \mathbf{K}_\lambda$. For every $\eta \in 2^\omega$, let $p_\eta \in \mathbf{S}^{na}(N)$ be an upper bound of $\langle p_{\eta\restriction_n} : n < \omega\rangle$ given by Fact 2.10. Observe that if $\eta \neq \nu \in 2^\omega$, $p_\eta \neq p_\nu$. Indeed, let n be the minimum n such that $\eta \restriction_n = \nu \restriction_n$ and $\eta(n) \neq \nu(n)$. Then $p_\eta \restriction_{M_{n+1}} = p_{\eta\restriction_n{}^\frown\eta(n)} \neq p_{\nu\restriction_n{}^\frown\nu(n)} = p_\nu \restriction_{M_{n+1}}$ by Condition (3) of the construction. Then $|\mathbf{S}^{na}(N)| \geq 2^{\aleph_0}$ which contradicts our assumption. $\square$

Remark 3.10. If $M \leq_{\mathbf{K}} N$, $p \in \mathbf{S}^{\lambda-unq}(M)$, $q \in \mathbf{S}^{\lambda-ext}(N)$ and $q \geq p$, then $q \in \mathbf{S}^{\lambda-unq}(N)$.

We are ready to prove one of the main results of the paper.

Theorem 3.11. Assume that $\mathbf{K}$ has amalgamation in λ, no maximal model in λ, and is stable for λ-algebraic types in λ. If $|\mathbf{S}^{na}(M)| < 2^{\aleph_0}$ for every $M \in \mathbf{K}_\lambda$ and $\mathbf{K}$ is $(< \lambda^+, \lambda)$-local, then $\mathbf{K}$ has a model of cardinality λ^{++}.

Proof. It is enough to show that $\mathbf{K}$ has no maximal models in λ^+.

Assume for the sake of contradiction that $M \in \mathbf{K}_{\lambda^+}$ is a maximal model. Let $N \leq_{\mathbf{K}} M$ such that $N \in \mathbf{K}_\lambda$. By the maximality of M together with Lemma 3.5, Lemma 3.9 and amalgamation in λ, there is $M_0 \in \mathbf{K}_\lambda$ with $N \leq_{\mathbf{K}} M_0 \leq_{\mathbf{K}} M$ and $q_0 \in \mathbf{S}^{\lambda-unq}(M_0)$. Let $\langle M_i \in \mathbf{K}_\lambda : i < \lambda^+\rangle$ be a resolution of M with M_0 as before. We build $\langle p_i : i < \lambda^+\rangle$ such that:

(1) $p_0 = q_0$;
(2) if $i < j < \lambda^+$, then $p_i \leq p_j$;
(3) for every $i < \lambda^+$, $p_i \in \mathbf{S}^{\lambda-unq}(M_i)$;
(4) for every $j < \lambda^+$, $\langle p_i : i < j\rangle$ is coherent .

$\boxed{\text{Construction}}$ The base step is given and the successor step can be achieved using Lemma 3.7 and Remark 3.10. So assume i is limit, take p_i to be an upper bound of $\langle p_j : j < i\rangle$ given by Fact 2.10. By Fact 2.10 $\langle p_j : j < i{+}1\rangle$ is coherent so we only need to show that $p_i \in \mathbf{S}^{\lambda-unq}(\bigcup_{j<i} M_j)$.

By Remark 3.10 it suffices to show that $p_i \in \mathbf{S}^{\lambda-ext}(\bigcup_{j<i} M_j)$. Since $p_0 \in \mathbf{S}^{\lambda-unq}(M_0)$ and $M_0 \leq_{\mathbf{K}} \bigcup_{j<i} M_j$, there is $q \in \mathbf{S}^{\lambda-ext}(\bigcup_{j<i} M_j)$ such that $q \geq p_0$ by Lemma 3.7.

We show that for every $j < i$, $q \restriction_{M_j} = p_i \restriction_{M_j}$. Let $j < i$. Since $q \restriction_{M_j} \in \mathbf{S}^{\lambda-ext}(M_j)$, $p_i \restriction_{M_j} = p_j \in \mathbf{S}^{\lambda-ext}(M_j)$ and both extend p_0 a λ-unique type, $q \restriction_{M_j} = p_i \restriction_{M_j}$.

Therefore, $q = p_i$ as $\mathbf{K}$ is $(< \lambda^+, \lambda)$-local. Hence $p_i \in \mathbf{S}^{\lambda-ext}(\bigcup_{j<i} M_j)$ as $q \in \mathbf{S}^{\lambda-ext}(\bigcup_{j<i} M_j)$.

$\boxed{\text{Enough}}$ Let $q^* \in \mathbf{S}^{na}(M)$ be an upper bound of the coherent sequence $\langle p_i : i < \lambda^+\rangle$ given by Fact 2.10. As q^* is a non-algebraic type, M has a proper extension which contradicts our assumption that M is maximal. $\square$

We use the previous theorem to obtain two corollaries with more natural assumptions. The next result is the result mentioned in the abstract.

Theorem 3.12. Suppose $\lambda < 2^{\aleph_0}$. Let $\mathbf{K}$ be an abstract elementary class with $\lambda \geq \mathrm{LS}(\mathbf{K})$. Assume $\mathbf{K}$ has amalgamation in λ, no maximal model in λ, and is stable in λ. If $\mathbf{K}$ is $(< \lambda^+, \lambda)$-local, then $\mathbf{K}$ has a model of cardinality λ^{++}.

Proof. We show that for every $M \in \mathbf{K}_\lambda$, $|\mathbf{S}^{na}(M)| < 2^{\aleph_0}$. This is enough by Theorem 3.11. Let $M \in \mathbf{K}_\lambda$. $|\mathbf{S}^{na}(M)| \leq \lambda$ by stability in λ. Since $\lambda < 2^{\aleph_0}$ by assumption, $|\mathbf{S}^{na}(M)| < 2^{\aleph_0}$. $\square$

Remark 3.13. For AECs $\mathbf{K}$ with $\mathrm{LS}(\mathbf{K}) = \aleph_0$ and $\lambda = \aleph_0$, the assumption that $\lambda < 2^{\aleph_0}$ is vacuous. This result for $(< \aleph_0, \aleph_0)$-tame AECs can be obtained using [ShVas18, 4.7], [ShVas18, 5.8], [Sh:h, II.4.13], but the result has never been stated in the literature. Moreover, the argument presented in this paper is significantly simpler than the argument using the results of Shelah and Vasey. Furthermore, the result is new for $(\aleph_0, \aleph_0)$-local AECs.

We can also weaken the stability assumption to stability for λ-algebraic types at the cost of strengthening the cardinal arithmetic hypothesis from $\lambda < 2^{\aleph_0}$ to $\lambda^+ < 2^{\aleph_0}$.

Lemma 3.14. Suppose $\lambda^+ < 2^{\aleph_0}$. Assume $\mathbf{K}$ has amalgamation in λ, no maximal model in λ, and is stable for λ-algebraic types in λ. If $\mathbf{K}$ is $(< \lambda^+, \lambda)$-local, then $\mathbf{K}$ has a model of cardinality λ^{++}.

Proof. Assume for the sake of contradiction that $\mathbf{K}_{\lambda^{++}} = \emptyset$. We show that for every $M \in \mathbf{K}_\lambda$, $|\mathbf{S}^{na}(M)| < 2^{\aleph_0}$. This is enough by Theorem 3.11.

Let $M \in \mathbf{K}_\lambda$. Then there is $M \leq_{\mathbf{K}} N \in \mathbf{K}_{\lambda^+}$ maximal. Every $p \in \mathbf{S}^{na}(M)$ is realized in N by amalgamation in λ and maximality of N. Thus $|\mathbf{S}^{na}(M)| \leq \|N\| = \lambda^+$. Since $\lambda^+ < 2^{\aleph_0}$ by assumption, $|\mathbf{S}^{na}(M)| < 2^{\aleph_0}$. $\square$

4. Additional results using an unproven claim of Shelah

In this section we present a natural assumption under which an AEC is stable for λ-algebraic types. We use this result together with the results of the previous section to give a positive answer to Grossberg's question for small cardinals assuming a mild locality condition for Galois types and without any stability assumptions. *All* the main results of this section rely on a result of Shelah [Sh:h, VI.2.11.(2)] (Fact 4.5 of this paper) for which Shelah does not provide an argument, for which the *standard* argument does not seem to work, and which we were unable to verify. See Remark 4.6 for more details.

The following couple of notions appear in [Sh:h, §VI].

Definition 4.1. $\mathbf{S}_*$ is $\leq_{\mathbf{K}_\lambda}$-*type-kind* when:

(1) $\mathbf{S}_*$ is a function with domain $\mathbf{K}_\lambda$.
(2) $\mathbf{S}_*(M) \subseteq \mathbf{S}^{na}(M)$ for every $M \in \mathbf{K}_\lambda$.
(3) $\mathbf{S}_*(M)$ commutes with isomorphisms for every $M \in \mathbf{K}_\lambda$.

Definition 4.2. $\mathbf{S}_1$ is *hereditarily in* $\mathbf{S}_2$ when: for $M \leq_{\mathbf{K}} N \in \mathbf{K}_\lambda$ and $p \in \mathbf{S}_2(N)$ we have that if $p \restriction_M \in \mathbf{S}_1(M)$ then $p \in \mathbf{S}_1(N)$. If $\mathbf{S}_2 = \mathbf{S}^{na}$ we will say that $\mathbf{S}_1$ is *hereditary.*

The proof of following proposition is straightforward.

Proposition 4.3. Assume $\mathbf{K}$ has amalgamation and no maximal model in λ. $\mathbf{S}^{\lambda-al}$ is $\leq_{\mathbf{K}_\lambda}$-type-kind and hereditary.

Definition 4.4. For $M \in \mathbf{K}$ and $\Gamma \subseteq \mathbf{S}^{na}(M)$, Γ is $\mathbf{S}_*$-*inevitable* if for every $N >_{\mathbf{K}} M$, if there is $p \in \mathbf{S}_*(M)$ realized in N then there is $q \in \Gamma$ realized in N.

The following result appears in [Sh:h] without a proof.

Fact 4.5 ([Sh:h, VI.2.11.(2)]). Assume $\mathbf{K}$ has amalgamation and no maximal model in λ. If

(1) $\mathbf{S}_*$ is $\leq_{\mathbf{K}_\lambda}$-type-kind and hereditary, and
(2) for every $N \in \mathbf{K}_\lambda$ there is an $\mathbf{S}_*$-inevitable $\Gamma_N \subseteq \mathbf{S}^{na}(N)$ of cardinality $\leq \lambda$,

then for every $M \in \mathbf{K}_\lambda$ we have that $|\mathbf{S}_*(M)| \leq \lambda$.

Remark 4.6. The previous result is the one mentioned in the introduction of this section. As mentioned there, the *standard* argument does not seem to work and we were unable to verify the result. The standard argument we are referencing here is the one used to show stability from the existence of a good λ-frame [Sh:h, II.4.2] . The reason that argument does not work is because we do not have any trace of local character. It is worth mentioning that the following two generalizations [JaSh13, 2.5.8] and [Vas, A.11] of that argument do not work either.

We give additional details on why the standard argument does not work, hoping that this could help elucidate the situation and eventually help prove the result. In the standard argument when $\mathbf{S}_* = \mathbf{S}^{na}$, one builds $\langle M_i \in K_\lambda : i \leq \lambda \rangle$ increasing and continuous with each M_{i+1} realizing Γ_{M_i}, hoping that eventually we realize all types in $\mathbf{S}_*(M_0)$ in M_λ. To show any type $\mathbf{gtp}(a/M_0, N) \in \mathbf{S}_*(M_0)$ is realized, one builds $\langle N_i : i < \lambda^+ \rangle$ and $\mathbf{K}$-embeddings $f_i : M_i \to N_i$ and shows that f_λ is an isomorphism, and hence $f_\lambda^{-1} \restriction_{N_0} : N_0 \to M_\lambda$ is enough. If f_λ is not an isomorphism, some type in $\mathbf{S}^{na}(M_\lambda)$ and hence some type in Γ_{M_λ} is realized in N_λ, and using local character, that realization can be "resolved" at some stage $i < \lambda$. Adapting this naively to the case when $\mathbf{S}_*$ is not necessarily $\mathbf{S}^{na}$, we expect that no type in Γ_{M_λ} and hence no type in $\mathbf{S}_*(M_\lambda)$ is realized in N_λ via f_λ. Without local character we cannot realize Γ_{M_λ} in earlier stages.

However this is more than what is needed and might be unnecessary, as this would imply that no $\mathbf{S}_*(M_\lambda)$ is realized in N_λ, while we only need that there is no extension of any type in $\mathbf{S}_*(M_0)$ to $\mathbf{S}^{na}(M_\lambda)$ (the type is in $\mathbf{S}_*(M_\lambda)$ since $\mathbf{S}_*$ is hereditary) is realized in $|N_\lambda| - |M_\lambda|$. Also, for our purpose, it would be enough if Fact 4.5 could be proved under the assumptions of Theorem 4.10.

We recall one last definition from [Sh:h, §VI].

Definition 4.7. Let $M \in \mathbf{K}_\lambda$. $p \in \mathbf{S}^{na}(M)$ is a $< \lambda^+$*-minimal type* if for every $M \leq_{\mathbf{K}} N \in \mathbf{K}_\lambda$, $|\{q \in \mathbf{S}^{na}(N) : q \restriction_M = p\}| \leq \lambda$. Let $\mathbf{S}^{<\lambda^+-min}(M)$ denote the $< \lambda^+$-minimal types over M.

Lemma 4.8. Assume $\mathbf{K}$ has amalgamation in λ. For every $M \in \mathbf{K}_\lambda$, $\mathbf{S}^{\lambda-al}(M) \subseteq \mathbf{S}^{<\lambda^+-min}(M)$.

Proof. Fix $M \in \mathbf{K}_\lambda$. We show the result by contrapositive. Let $p \in \mathbf{S}^{na}(M) - \mathbf{S}^{<\lambda^+-min}(M)$, i.e., p has at least λ^+ extensions to $\mathbf{S}^{na}(N)$ for some $M \leq_{\mathbf{K}} N \in \mathbf{K}_\lambda$. Using the amalgamation property in λ one can construct $M^* \in \mathbf{K}_{\lambda^+}$ such that $M \leq_{\mathbf{K}} N \leq_{\mathbf{K}} M^*$ and M^* realizes λ^+ many extensions of p to $\mathbf{S}^{na}(N)$. In particular, M^* has λ^+ realizations of p. Hence p has the λ-extension property by Fact 3.2. $\square$

Fact 4.9 ([Sh:h, VI.2.18])**.** ($2^\lambda < 2^{\lambda^+}$) Assume $\mathbf{K}$ has amalgamation and no maximal model in λ. If

(1) $\mathbf{S}_*$ is $\leq_{\mathbf{K}_\lambda}$-type-kind and hereditary,
(2) $\mathbf{S}_* \subseteq \mathbf{S}^{<\lambda^+-min}$, and
(3) there is $M \in \mathbf{K}_\lambda$ such that:
    (a) $|\mathbf{S}_*(M)| \geq \lambda^+$, and
    (b) if $M \leq_{\mathbf{K}} N \in \mathbf{K}_\lambda$, no subset of $\mathbf{S}_*(N)$ of size $\leq \lambda$ is $\mathbf{S}_*$-inevitable,

then $\mathbb{I}(\mathbf{K}, \lambda^+) = 2^{\lambda^+}$.

We show how to get stability for λ-algebraic types. This result appears in [Yan], we include a proof of the result as it is one of the key components of our argument and that paper is still under review.

Theorem 4.10. ($2^\lambda < 2^{\lambda^+}$) If $\mathbb{I}(\mathbf{K}, \lambda) = 1 \leq \mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$, then $\mathbf{K}$ is stable for λ-algebraic types in λ.

Proof. Assume for the sake of contradiction that there is $M \in \mathbf{K}_\lambda$ such that $|\mathbf{S}^{\lambda-al}(M)| \geq \lambda^+$. Observe that $\mathbf{K}$ has amalgamation and no maximal models in λ by Fact 2.2.

We show that conditions (1) to (3) of Fact 4.9 hold for $\mathbf{S}_* = \mathbf{S}^{\lambda-al}$. This is enough as Fact 4.9 implies that $\mathbb{I}(\mathbf{K}, \lambda^+) = 2^{\lambda^+}$ and we assumed that $\mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$. Condition (1) is Proposition 4.3, Condition (2) is Lemma 4.8 and Condition (3).(a) is our assumption that $|\mathbf{S}^{\lambda-al}(M)| \geq \lambda^+$. So we only need to show Condition (3).(b). Let $M \leq_{\mathbf{K}} N \in \mathbf{K}_\lambda$.

Claim: There is no $\Gamma \subseteq \mathbf{S}^{\lambda-al}(N)$ such that $|\Gamma| \leq \lambda$ and Γ is $\mathbf{S}^{\lambda-al}$-inevitable.

Proof of Claim: Otherwise, suppose there exists such Γ. If we show that Condition (2) of Fact 4.5 for $\mathbf{S}_* = \mathbf{S}^{\lambda-al}$ holds, we would be done as Fact 4.5 would imply that $|\mathbf{S}^{\lambda-al}(M)| \leq \lambda$ which contradicts the assumption that $|\mathbf{S}^{\lambda-al}(M)| \geq \lambda^+$. Let $L \in \mathbf{K}_\lambda$. Then there is $f : L \cong N$ an isomorphism by λ-categoricity. Using f we can copy Γ to a $\Gamma_L \subseteq \mathbf{S}^{na}(L)$ such that $|\Gamma_L| \leq \lambda$ and Γ_L is $\mathbf{S}^{\lambda-al}$-inevitable as Γ is $\mathbf{S}^{\lambda-al}$-inevitable. $\dagger_{\text{Claim}}$ $\square$

We obtain the following positive answer to Grossberg's question for small cardinals assuming a mild locality condition for Galois types and without any stability assumption.

Theorem 4.11. ($2^\lambda < 2^{\lambda^+}$) Suppose $\lambda^+ < 2^{\aleph_0}$ and Fact 4.5 holds. Assume $\mathbb{I}(\mathbf{K}, \lambda) = 1 \leq \mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$. If $\mathbf{K}$ is $(< \lambda^+, \lambda)$-local, then $\mathbf{K}$ has a model of cardinality λ^{++}.

Proof. It follows that $\mathbf{K}$ has amalgamation in λ by Fact 2.2 and it is clear that $\mathbf{K}$ has no maximal model in λ. Moreover, $\mathbf{K}$ is stable for λ-algebraic types in λ by Theorem 4.10. Therefore, $\mathbf{K}$ has a model of cardinality λ^{++} by Lemma 3.14. □

Remark 4.12. For AECs $\mathbf{K}$ with $\mathrm{LS}(\mathbf{K}) = \aleph_0$ and $\lambda = \aleph_0$. The previous result is known for universal classes (even without the assumption that $2^{\aleph_0} > \aleph_1$) [MaVa18, 3.3], but it is new for $(< \aleph_0, \aleph_0)$-tame AECs. For $\lambda > \aleph_1$, the result is new even for universal classes.

Remark 4.13. The set theoretic assumption that $\lambda^+ < 2^{\aleph_0}$ can be replaced by the model theoretic assumption that $|\mathbf{S}^{na}(M)| < 2^{\aleph_0}$ for every $M \in \mathbf{K}_\lambda$ by using Theorem 3.11 instead of Lemma 3.14.

Let us consider the following property on chains of types:

Definition 4.14. A type family $\mathbf{S}_*$ is λ-compact if for every limit ordinal $\delta < \lambda^+$, for every $\langle M_i \in \mathbf{K}_\lambda : i < \delta\rangle$ an increasing continuous chain and for every coherent sequence of types $\langle p_i \in \mathbf{S}_*(M_i) : i < \delta\rangle$, there is an upper bound $p \in \mathbf{S}_*(\bigcup_{i<\delta} M_i)$ to the sequence such that $\langle p_i \in \mathbf{S}_*(M_i) : i < \delta + 1\rangle$ is a coherent sequence.

Remark 4.15. For every $M \in \mathbf{K}_\lambda$, let

$$\mathbf{S}^{\lambda-sunq}(M) = \{p \in \mathbf{S}^{\lambda-unq}(M) : p \text{ has the } \lambda\text{-strong extension property}\}.$$

The limit step of Theorem 3.11 basically shows that if $\mathbf{K}$ is $(< \lambda^+, \lambda)$-local then $\mathbf{S}^{\lambda-sunq}$ is λ-compact.

The locality assumption on types and cardinal arithmetic assumption that $\lambda^+ < 2^{\aleph_0}$ can be dropped from Theorem 4.11 if instead we assume that the larger class of types $\mathbf{S}^{\lambda-ext}$ is λ-compact. The result still uses Fact 4.5.

Corollary 4.16. ($2^\lambda < 2^{\lambda^+}$) Assume $1 = \mathbb{I}(\mathbf{K}, \lambda) \leq \mathbb{I}(\mathbf{K}, \lambda^+) < 2^\lambda$, and. If $\mathbf{S}^{\lambda-ext}$ is λ-compact, then $\mathbf{K}$ has a model of cardinality λ^{++}.

Proof. The proof is similar to that of Theorem 3.11, except that in the construction we only require that p_i has the λ-extension property instead of being a λ-unique type. At limit stage we can do the construction using that the types only have the λ-extension property because of the assumption that $\mathbf{S}^{\lambda-ext}$ is λ-compact. □

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